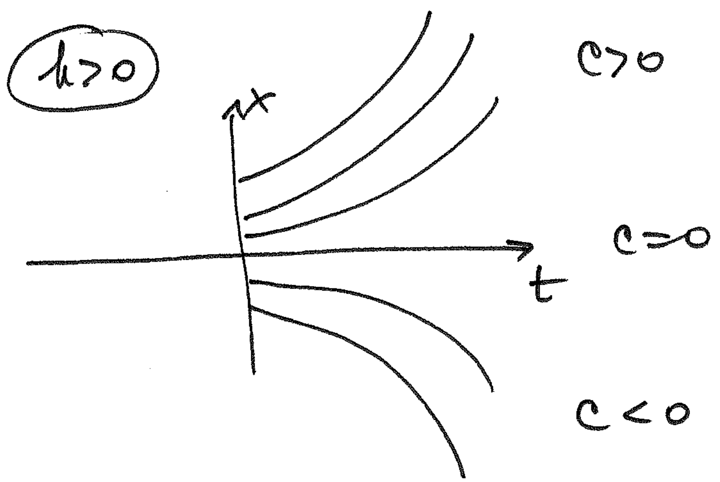
Lecture 5

Ex $\frac{dx}{dt} = kx$, $x(0) = x_0$

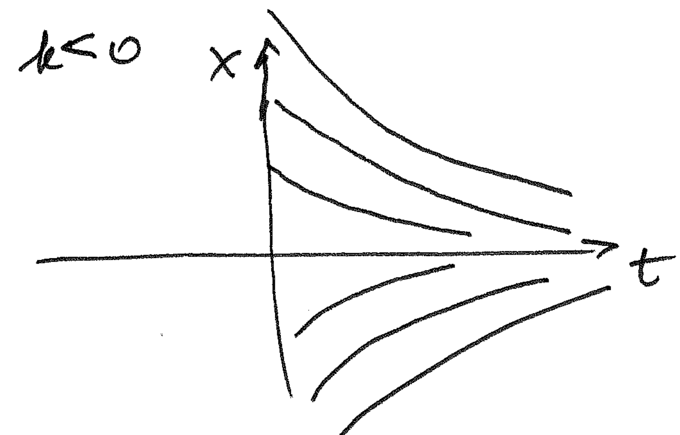
at $t=0$ $\Rightarrow$ $\underbrace{x(0)}_{= x_0} = C \underbrace{e^{k \cdot 0}}_{=1}$ $\Rightarrow$ $\boxed{C = x_0}$ $\Rightarrow$ $\boxed{x(t) = x_0 e^{kt}}$ is a solution of $\frac{dx}{dt} = kx$, $x(0) = x_0$

$k>0$



exponential growth

$k<0$



exponential decay

Population Model

$P(t)$: # of individuals in a population

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assume :

$\beta > 0$: constant birth rate } per population per unit of time

$\delta > 0$: const death rate } per population per unit of time

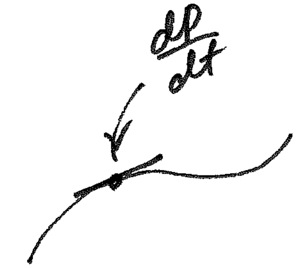
Δt : small interval of time

$\beta P(t) \Delta t$: # of individuals added to $P(t)$ due to birth (# of birth)

$\delta P(t) \Delta t$: # of death

$\Delta P = \beta P(t) \Delta t - \delta P(t) \Delta t = (\beta - \delta) P(t) \Delta t$

change in P during Δt



Then $\frac{\Delta P}{\Delta t} = \underbrace{(\beta - \delta)}_{= k} P$ $\quad k = \beta - \delta$

$$\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = \frac{dP}{dt} = kP \Rightarrow \boxed{\frac{dP}{dt} = kP}$$

natural growth/decay DE

rate of change of P wrt t is proportional to P

Solution : $P(t) = P(0) e^{kt}$

Radioactive Decay

$N(t)$: # of atoms of a certain radioactive isotope in a sample

Observations: atoms (isotope) decay by the ~~fr~~ same fraction. They decay to other isotope of the same material or isotopes of other material.

$N(t)$ behaves like a population model w/ constant birth/death rates but only has decay rate ($\beta = 0$, $\delta > 0$)

$$\frac{dN}{dt} = (\cancel{\beta}^{\,0} - \delta) N \Rightarrow \frac{dN}{dt} = -\delta N, \quad \delta > 0$$

Typically we would write

$$\frac{dN}{dt} = -kN, \quad k > 0$$

For ^{14}C, $k \approx 0.0001216$, t: years

Def The half-life τ of a radioactive isotope is time required for a half of a sample to decay.

$$\frac{dN}{dt} = -kN, \qquad N(0) = N_0 \ : \ \text{initial \# of isotopes}$$

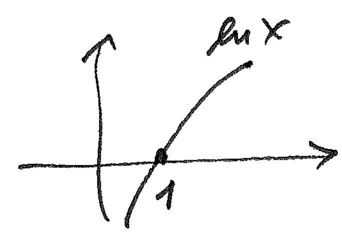
at $t = \tau$, $N(\tau) = \frac{1}{2} N_0$

$\frac{dN}{dt} = -kN$ has solution $N(t) = Ce^{-kt}$

at $t=0$, $\underbrace{N(0)}_{= N_0} = C\underbrace{e^{-k\cdot 0}}_{=1} \Rightarrow C = N_0 \Rightarrow \boxed{N(t) = N_0 e^{-kt}}$

at $t = \tau$: $\underbrace{N(\tau)}_{\frac{1}{2}N_0} = N_0 e^{-k\tau} \Rightarrow \frac{1}{2}\cancel{N_0} = \cancel{N_0} e^{-k\tau}$

$\boxed{\ln\frac{a}{b} = \ln a - \ln b}$



ln $\Rightarrow \boxed{e^{-k\tau} = \frac{1}{2}} \Rightarrow -k\tau = \ln\frac{1}{2}$

$-k\tau = \cancel{\ln 1}^{0} - \ln 2$

$k\tau = \ln 2$

$\therefore \boxed{\tau = \frac{\ln 2}{k}}$: half-life time

For ^{14}C, $\tau = \frac{\ln 2}{k} = \frac{\ln 2}{0.0001216} \approx 5700$ years

Ex Carbon extracted from an ancient skull contained only $\frac{1}{6}$ as much as ^{14}C as carbon extracted from present days bones. How old is the skull? The half-life of ^{14}C is approximately 5700 years.

Solution

$\tau = 5700$ years $\Rightarrow$ we can find $\boxed{k = \frac{\ln 2}{\tau}}$

$N(t)$: amount of ^{14}C

$N(0) = N_0$

at $\underbrace{t = t^*}_{\text{age}}$, $N(t^*) = \frac{1}{6} N_0$

$\frac{dN}{dt} = -kN$, $N(0) = N_0 \Rightarrow N(t) = N_0 e^{-kt}$

at $t = t^*$: $N(t^*) = N_0 e^{-k \cdot t^*}$

$$\frac{1}{6}\cancel{N_0} = \cancel{N_0}\, e^{-kt^*} \Rightarrow \underbrace{e^{-kt^*}}_{=\frac{1}{e^{kt^*}}} = \frac{1}{6}$$

we can solve for t^*

here $k = \frac{\ln 2}{\tau}$

$$e^{kt^*} = 6 \quad \Big| \ln$$

$$kt^* = \ln 6 \Rightarrow t^* = \frac{\ln 6}{k} = \frac{\ln 6}{\frac{\ln 2}{\tau}} = \frac{\ln 6 \cdot 5700}{\ln 2} \approx 14{,}735 \text{ years}$$