Lecture 6

Equilibrium solutions and stability (More in S 2.2)

Def a DE $y' = f(y)$ (autonomous DE) has a constant solution $y(t) \equiv c$ if and only if $f(c) = 0$. In this case, the constant solution c is called an equilibrium solution of the DE $y' = f(y)$.

Ex $y' = \underbrace{y}_{f(y)}$ has 1 equilibrium solution $c = 0$ or $y(t) \equiv 0$

$y' = \underbrace{y^2 - 1}_{f(y)}$ has 2 equil. solutions $c = \pm 1$ or $y(t) \equiv \pm 1$

Def An equilibrium solution c is stable if $y(t) \to c$ as $t \to \infty$ for all nearby solutions. Otherwise, the equilibrium solution is unstable.

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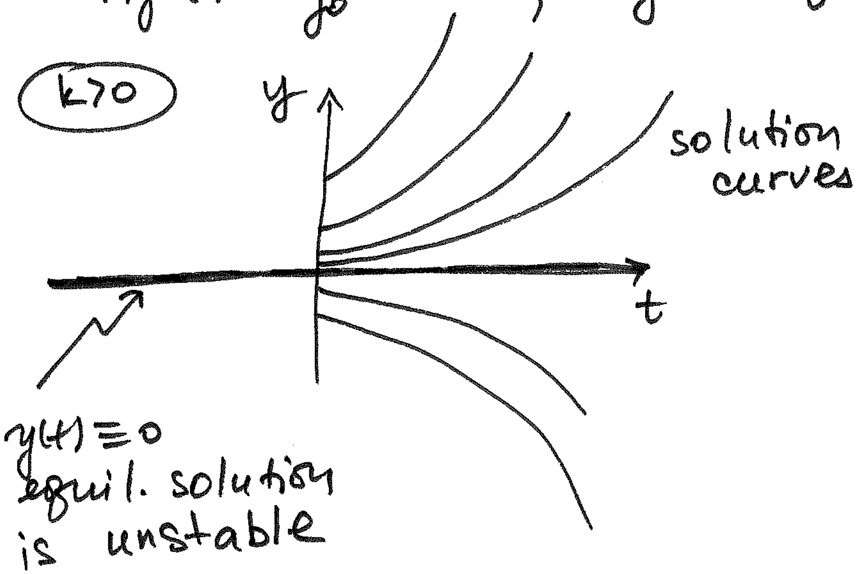
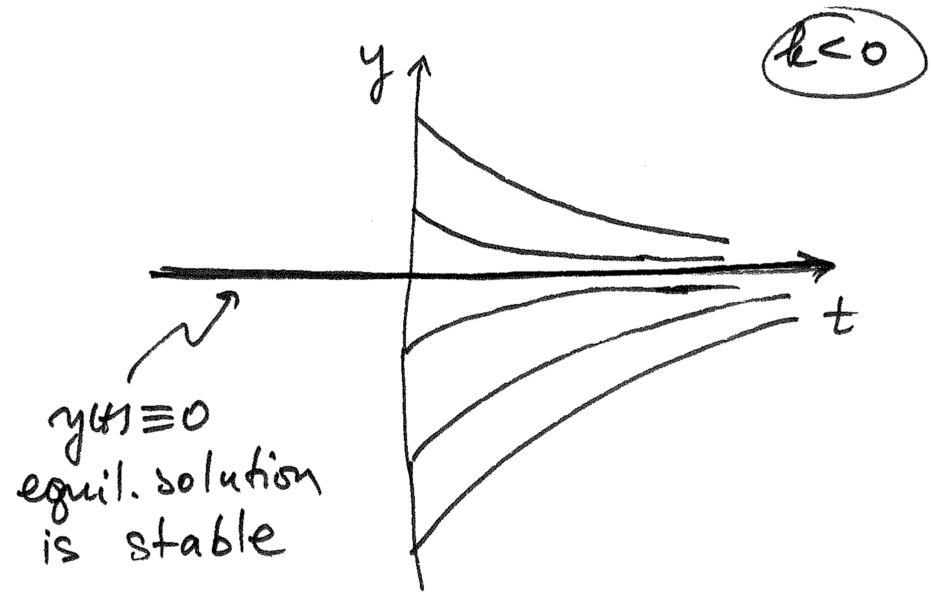
$\underline{\underline{\text{Ex}}}$ $y' = \underbrace{ky}_{f(y)}$ has 1 equil. solution $y(t) \equiv 0$

Is it stable or unstable?

$y(t) = y_0 e^{kt}$, $y(0) = y_0$

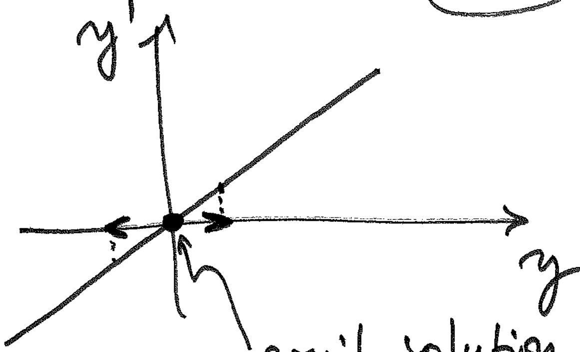




Phase Plane

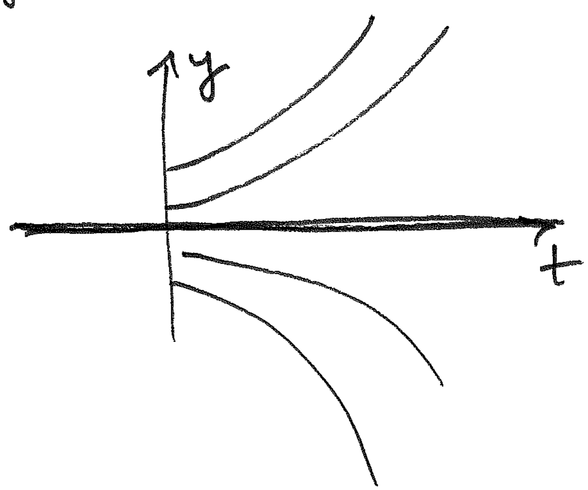
$y' = ky$

$k>0$

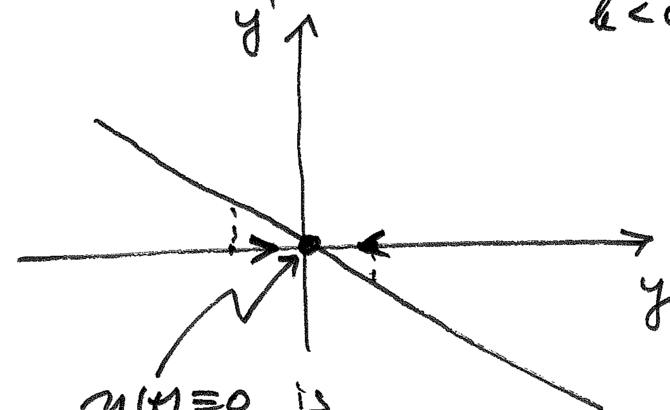


equil. solution $y(t)=0$

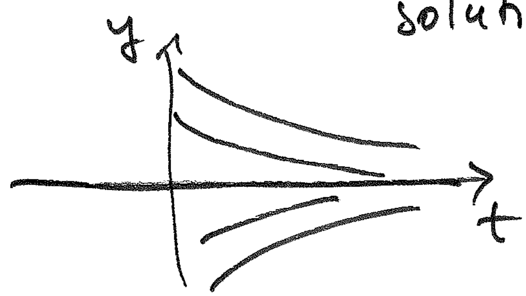
$y=0$ is unstable



$k<0$



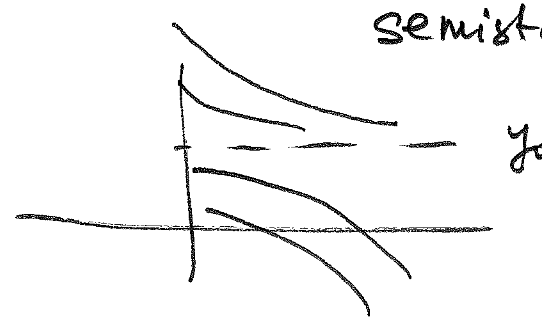
$y(t)\equiv 0$ is stable equil. solution



Aside

y_0

semistable



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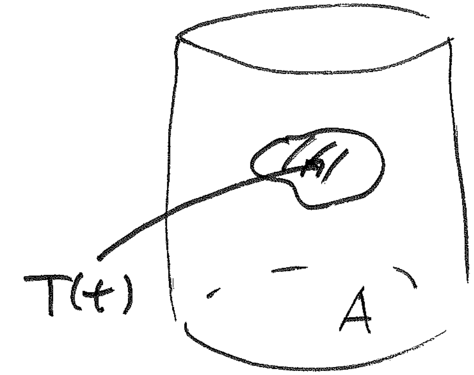
Newton's Law of Cooling / Heating

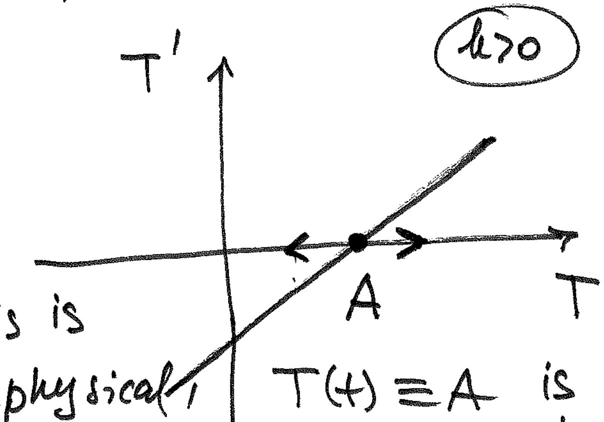
$T(t)$: temperature of a body immersed in medium of temperature A



$$\frac{dT}{dt} = k(T - A)$$

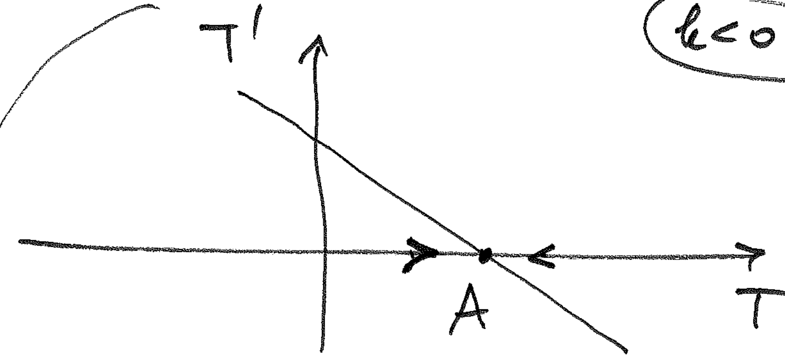
rate of change of T is proportional to difference of temp. $T(t)$ of a body and temperature A of surrounding medium

$T' = k(T - A)$ has 1 equil. solution $T(t) \equiv A$

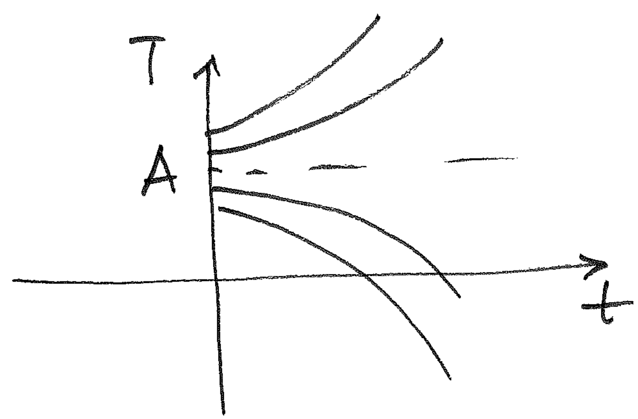


This is unphysical!

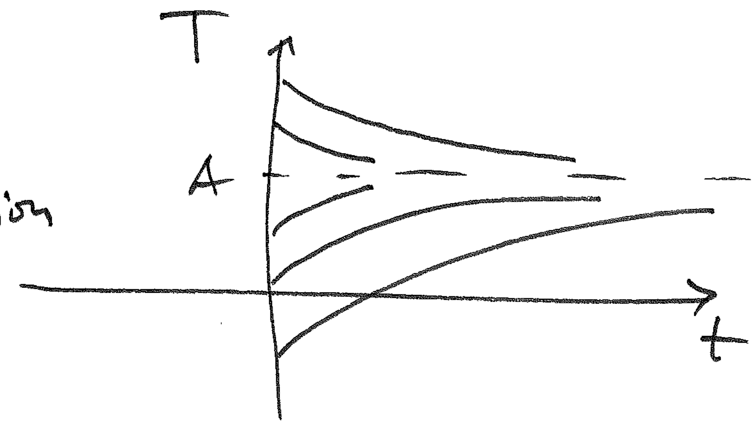
$T(t) \equiv A$ is unstable equil. solution



$T(t) \equiv A$ is stable equil. solutions



$T(t) \equiv A$ is equil. solution



Now we solve

$$\frac{dT}{dt} = -k(T-A), \qquad k>0$$

$$\frac{dT}{T-A} = -k\,dt$$

$$\int \frac{dT}{T-A} = -k\int dt$$

$$\ln|T-A| = -kt + \tilde{c}$$

$$|T-A| = e^{-kt+\tilde{c}}$$

$$\boxed{T-A = Ce^{-kt}}$$

IC: $T(0) = T_0$

$$\underbrace{T(0)}_{=T_0} - A = C\underbrace{e^{-k\cdot 0}}_{=1} \Rightarrow C = T_0 - A$$

$$\boxed{T(t) = A + (T_0 - A)e^{-kt}}$$ solution of $T' = -k(T-A)$, $T(0) = T_0$

Ex A pitcher of buttermilk initially at 25°C is cooled on the front porch where the temperature is 5°C. Suppose the temperature dropped to 18°C in 10 min. When will it be 10°C?

$T(0) = 25°C$

$A = 5°C$

$T(10) = 18°C$

Find $t = t^*$: $T(t^*) = 10°C$.

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Solution: $T(t) = A + (T_0 - A)e^{-kt}$

$$T(t) = 5 + (25-5)e^{-kt} = \boxed{5 + 20e^{-kt} = T(t)}$$

To find k, we use $T(10) = 18$

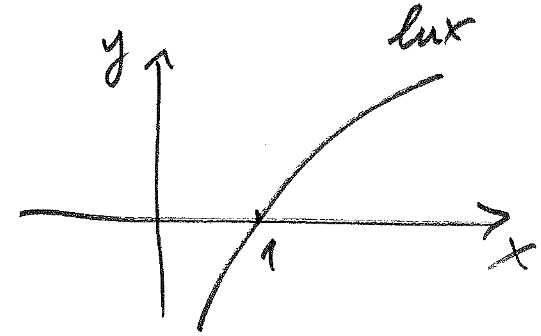
$$5 + 20e^{-k\cdot 10} = 18$$

$$20e^{-10k} = 13$$

$$e^{-10k} = \frac{13}{20} \quad \Big|\ \ln$$

$$-10k = \ln\frac{13}{20} \Rightarrow k = -\frac{\overbrace{\ln(13/20)}^{<0}}{10} > 0$$

or $\boxed{k = \frac{\ln(20/13)}{10}}$



$$\ln\frac{a}{b} = \ln a - \ln b = -(\ln b - \ln a) = -\ln\frac{b}{a}$$

To find $t = t^*$ when $T(t^*) = 10$,
we solve

$$\underbrace{T(t^*)}_{\substack{\shortparallel \\ 10}} = 5 + 20e^{-k t^*} \quad \text{for } t^*$$

known

$$10 = 5 + 20 e^{-kt^*}$$

$$5 = 20 e^{-kt^*}$$

$$\underbrace{e^{-kt^*}}_{= \frac{1}{e^{kt^*}}} = \frac{1}{4} \quad \text{or} \quad e^{kt^*} = 4 \Rightarrow kt^* = \ln 4$$

$$t^* = \frac{\ln 4}{k} = \frac{\overbrace{\ln 4 \cdot 10}^{>0}}{\underbrace{\ln \frac{20}{13}}_{>0}} > 0$$

$$k = \frac{\ln \frac{20}{13}}{10}$$

$$t^* \approx 32.18 \text{ min}$$