Lecture 7

1.5 First Order Linear DEs

General 1^{st} order linear DE is

$$\frac{1}{a_1(x)} \Big| \quad a_1(x)\frac{dy}{dx} + a_0(x)\,y = R(x), \qquad a_1 \neq 0 \qquad (1)$$

Aside

$y = ax + b$: linear

$y = ax^2 + bx + c$: quadratic

Canonical form:

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

where $P(x) = \dfrac{a_0(x)}{a_1(x)}$, $\quad Q(x) = \dfrac{R(x)}{a_1(x)}$

Def A linear DE (1) is homogeneous if $R(x) \equiv 0$.
Otherwise, this DE is nonhomogeneous.

Method of Integrating Factor

Consider

$$\frac{dy}{dx} + P(x)\,y = Q(x) \qquad (2)$$

Let $\boxed{\rho(x) = e^{\int P(x)\,dx}}$: integrating factor

Note

$$\frac{d\rho}{dx} = e^{\int P(x)\,dx} \cdot P(x)$$

Multiply both sides of (2) by $e^{\int P(x)\,dx} = \rho(x)$

$$\underbrace{e^{\int P(x)\,dx}}_{\rho(x)} \cdot \frac{dy}{dx} + \underbrace{e^{\int P(x)\,dx} \cdot P(x)}_{\frac{d\rho}{dx}}\,y = \underbrace{e^{\int P(x)\,dx}}_{\rho(x)} \cdot Q(x)$$

Note

$$\frac{d}{dx}\left(\rho(x)y(x)\right) = \rho'(x)\cdot y + \rho(x)\cdot y'$$

Hence

$$\frac{d}{dx}\left[\rho(x)y(x)\right] = \rho(x)\cdot Q(x)$$

Solve for $y(x)$. Integrate first.

$$\rho(x)y(x) = \int \rho(x)Q(x)\,dx + C \quad \Big| \ \frac{1}{\rho(x)}$$

Then

$$y(x) = \frac{1}{\rho(x)}\left[\int \rho(x)Q(x)\,dx + C\right]$$: do not memorize

But you should remember

$$\boxed{\begin{aligned} \rho(x) &= e^{\int P(x)\,dx} \\ \rho y &= \int \rho Q\,dx + C \end{aligned}}$$

: integrating factor

: solve for y

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Ex Solve

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\frac{dy}{dx} + \underbrace{\frac{2}{x}}_{P(x)} y = \underbrace{x^2}_{Q(x)}$$

$$\boxed{a \ln b = \ln b^a}$$

$$\rho(x) = e^{\int P(x)dx} = e^{\int \frac{2}{x} dx} = e^{2 \ln|x|} = e^{\ln x^2} = x^2$$

$$\rho \cdot y = \int \rho Q \, dx + C$$

$$x^2 \cdot y(x) = \int x^2 \cdot x^2 dx + C$$

$$x^2 \cdot y(x) = \int x^4 dx + C$$

$$x^2 \cdot y(x) = \frac{x^5}{5} + C \quad \Big| \, \frac{1}{x^2}$$

$$\boxed{y(x) = \frac{x^3}{5} + \frac{C}{x^2}}$$

Ex Solve

$$x \frac{dy}{dx} - 3y = x^4 e^x, \qquad x>0 \qquad \Big| \cdot \frac{1}{x}$$

not in canonical form since coeff of $\frac{dy}{dx}$ is not 1.

Divide both sides by x.

$$\frac{dy}{dx} \underbrace{- \frac{3}{x}}_{P(x)} y = \underbrace{x^3 e^x}_{Q(x)}$$

$$\rho(x) = e^{\int P(x)\,dx} = e^{\int \left(-\frac{3}{x}\right) dx} = e^{-3 \ln x} = e^{\ln x^{-3}} = x^{-3} = \frac{1}{x^3}$$

$$\rho y = \int \rho Q\,dx + C$$

$$\frac{1}{x^3} y = \int \frac{1}{\cancel{x^3}} \cancel{x^3} e^x\,dx + C$$

$$\frac{1}{x^3} y = \int e^x\,dx + C$$

$$\frac{1}{x^3} y = e^x + C \quad \Big| \cdot x^3$$

$$\boxed{y(x) = x^3 \left(e^x + C\right)}$$

Another form of the method of integrating factor.

$$\boxed{\rho(x) = e^{\int_{x_0}^{x} P(t)\,dt}}$$

$y(x_0) = y_0$: IC

$$\rho(x)\, y(x) = \int_{x_0}^{x} \rho(t)\, Q(t)\, dt + \underbrace{\rho(x_0)}_{= 1} \underbrace{y(x_0)}_{y_0}$$

$$\Rightarrow \boxed{\rho(x)\, y(x) = \int_{x_0}^{x} \rho(t)\, Q(t)\, dt + y_0}$$: then solve for $y(x)$

Mixtures

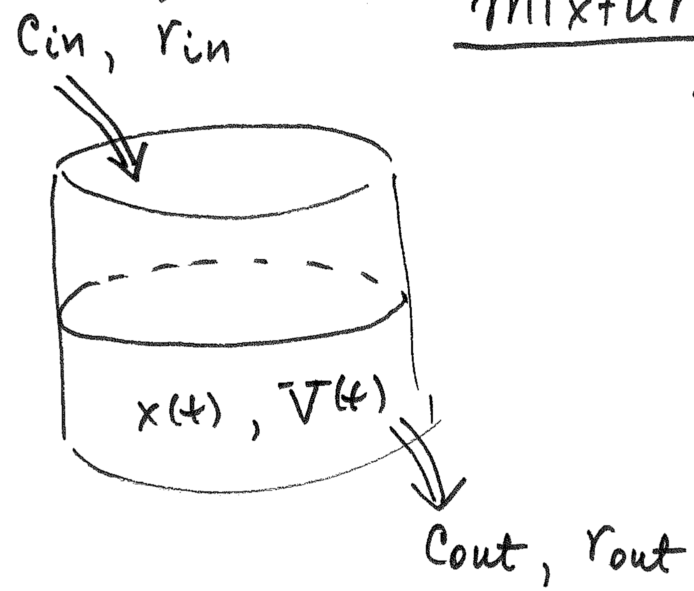
Problem A tank initially contains 60 gallons of brine (water + salt) in which 4 pounds of salt are dissolved. Brine containing 2 lbs of salt per gallon enters the tank at the rate of 3 gallons per minute and the well stirred mixture leaves the tank at the rate of 4 gallons per minute.

(a) How much salt is in the tank at any time? $x(t)$ - ?

(b) How much salt is in the tank at $t=5$?

(c) What is the salt concentration $c(t)$ at $t=0, 10, 70$?

Mixtures



$x(t)$: amount of salt in tank [lbs]

$V(t)$: volume of solution [gal]

$c(t)$: concentration of salt

$$c(t) = \frac{x(t)}{V(t)} \quad \left[\frac{\text{lbs}}{\text{gal}}\right]$$

C_{in} : concentration of solution entering the tank $\left[\frac{\text{lbs}}{\text{gal}}\right]$

r_{in} : rate at which solution enters the tank $\left[\frac{\text{gal}}{\text{min}}\right]$

$C_{out} = c(t) = \frac{x(t)}{V(t)}$: concentration of solution that leaves the tank $\left[\frac{\text{lbs}}{\text{gal}}\right]$

r_{out} : rate at which solution leaves the tank $\left[\frac{\text{gal}}{\text{min}}\right]$

#45 S1.1 Suppose a population P of rodents satisfies the diff. eq^{n} $\frac{dP}{dt} = kP^2$. Initially, there are $P(0)=2$ rodents, and their number is increasing at the rate of $\frac{dP}{dt} = 1$ rodent per month when there are $P=10$ rodents. How long will it take for this population to grow to a hundred rodents? To a thousand?

□ $\frac{dP}{dt} = kP^2$, solution is $P(t) = \frac{1}{C-kt}$ (from #43 or use separation of variables)

$P(0)=2 \Rightarrow$ at $t=0$ $\underbrace{P(0)}_{2} = \frac{1}{C-k\cdot 0} \Rightarrow \boxed{C=\frac{1}{2}}$

$\therefore P(t) = \frac{1}{\frac{1}{2}-kt}$

$\frac{dP}{dt} = kP^2 \Rightarrow \underbrace{\frac{dP}{dt}\Big|_{P=10}}_{1} = k\cdot 10^2 \Rightarrow \boxed{k=\frac{1}{100}} \Rightarrow P(t) = \frac{1}{\frac{1}{2}-\frac{1}{100}t} = \boxed{\frac{100}{50-t} = P(t)}$

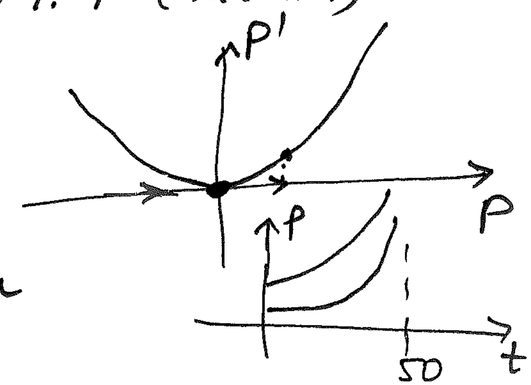
$P(t)=100$ when $\frac{100}{50-t} = 100 \Rightarrow 50-t=1$ or $t=49$ (month)

$P(t)=1000$ when $\frac{100}{50-t} = 1000$ or $50-t=\frac{1}{10} \Rightarrow t=49.9$ (month)

<u>Note</u> As $t \to 50$, $P(t) \to \infty$

$\frac{dP}{dt} = kP^2$

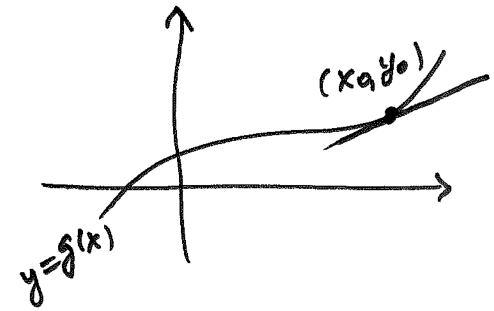
P=0 : unstable equil. solution



$P' > 0 \Rightarrow P \nearrow$ as $t \nearrow$

#31
S1.1

The line tangent to the graph of g at (x, y) passes through the point $(-y, x)$



tangent line $y - y_0 = k(x - x_0)$, k: slope

here slope $k = \frac{dy}{dx}$, $(x_0, y_0) = (-y, x)$

$\Rightarrow y - x = \frac{dy}{dx}(x + y)$ or $\boxed{\frac{dy}{dx} = \frac{y-x}{x+y}}$