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Lecture 8

$\frac{dx}{dt}$: rate of change of amount of salt with time

Basic principle

$\boxed{\frac{dx}{dt} = \text{amt of salt entering per minute} - \text{amt of salt leaving per min}}$

$$\boxed{\frac{dx}{dt} = c_{in} \cdot r_{in} - c_{out} \cdot r_{out}}$$

Units

$$\left[\frac{lbs}{min}\right] = \left[\frac{lbs}{gal}\right] \cdot \left[\frac{\cancel{gal}}{min}\right] - \left[\frac{lbs}{\cancel{gal}}\right] \cdot \left[\frac{\cancel{gal}}{min}\right]$$ ✓

We know

$$c_{out} = \frac{x(t)}{V(t)}$$

Volume is not constant here.

$$\boxed{\frac{dV}{dt} = r_{in} - r_{out}}$$

$$\left[\frac{gal}{min}\right] = \left[\frac{gal}{min}\right] - \left[\frac{gal}{min}\right]$$ ✓

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We can find $V(t)$.

In this problem: $V(0)=60$ gal, $x(0)=4$ lbs

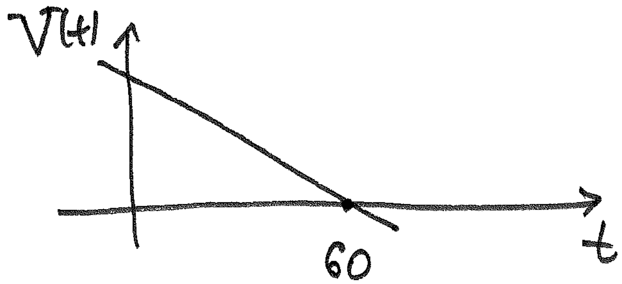
$c_{in} = 2$ lbs/gal, $r_{in} = 3$ gal/min, $r_{out} = 4$ gal/min

$\frac{dV}{dt} = 3-4 \Rightarrow \frac{dV}{dt} = -1 \Rightarrow V = -t + C$

$V(0)=60 \Rightarrow$ at $t=0$: $\underbrace{V(0)}_{=60} = -0 + C$

$\Rightarrow C = 60$

$\Rightarrow \boxed{V(t) = -t + 60}$



$\frac{x(t)}{V(t)} = \frac{x(t)}{-t+60}$

$\frac{dx}{dt} = c_{in} r_{in} - c_{out} r_{out}$

$\frac{dx}{dt} = 2 \cdot 3 - \frac{x}{60-t} \cdot 4$

or $\boxed{\frac{dx}{dt} + \frac{4}{60-t} x = 6}$ 1st order linear DE for $x(t)$

We can use the method of integrating factor to solve this DE.

Here $P(t) = \frac{4}{60-t}$ $\quad Q(t) = 6$

$$\frac{dx}{dt} + P(t)x = Q(t)$$
$$\rho(t) = e^{\int P(t)\,dt}$$
$$\rho \cdot x = \int \rho Q\,dt + C$$

$$\rho(t) = e^{\int P(t)\,dt} = e^{\int \frac{4}{60-t}\,dt} = e^{-4\ln|60-t|} =$$

$$= (60-t)^{-4} = \frac{1}{(60-t)^4}$$

$$\rho(t)\,x(t) = \int \rho(t)\,Q(t)\,dt + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\frac{1}{(60-t)^4}\,x(t) = \int \frac{1\cdot 6}{(60-t)^4}\,dt + C$$

$$\frac{1}{(60-t)^4}\cdot x(t) = 2(60-t)^{-3} + C \quad \Big| \cdot (60-t)^4$$

$$x^a \cdot x^b = x^{a+b}$$

We can solve for $x(t)$:

$$x(t) = 2(60-t) + C(60-t)^4$$

IC: $x(0) = 4 \Rightarrow$ at $t=0$: $4 = 2\cdot 60 + C\cdot 60^4 \Rightarrow \boxed{C = -\frac{116}{60^4}}$

(a) $\boxed{x(t) = 2(60-t) - 116\left(\frac{60-t}{60}\right)^4}$: amount of salt $x(t)$ at any time t

(b) amount of salt at $t=5$ is

$x(5) = 2(60-5) - 116\left(\frac{60-5}{60}\right)^4 = 28.1$ (lbs)

(c) concentration of salt:

$c(t) = \frac{x(t)}{V(t)} = \frac{1}{60-t}\left[2(60-t) - 116\left(\frac{60-t}{60}\right)^4\right]$

$V(t) = 60-t$ $\Bigg\}$ $\boxed{c(t) = 2 - \frac{116}{60}\left(\frac{60-t}{60}\right)^3}$ concentration of salt at any time t

$c(0) = 2 - \frac{116}{60} = $ or $\frac{x(0)}{V(0)} = \frac{4}{60} = 0.067$ lb/gal

$c(10) = 2 - \frac{116}{60}\left(\frac{60-10}{60}\right)^3 = 0.881$ lb/gal

$c(70)$ – ? undefined since tank is empty at $t=60$ min!

2.1 Populations

Logistic Equation

$P(t)$: population size

M: population threshold or maximum population size or carrying capacity of an environment (due to limited resources), $M>0$

$$\frac{dP}{dt} = \underbrace{k(M-P)}P, \quad k>0 \quad : \quad \text{logistic equation}$$

$k(M-P)$: variable growth rate

Aside: $\frac{dP}{dt} = kP$ – exp growth/decay

$k = \beta - \delta$ – const

$kM - kP$

birth rate — variable death rate

β: birth rate

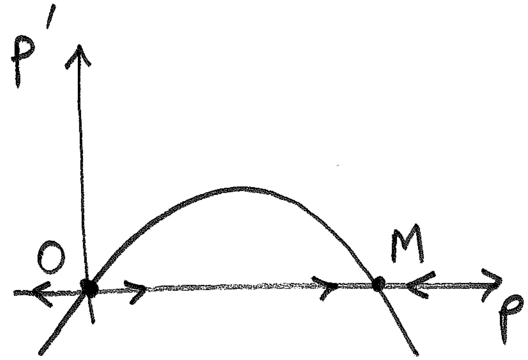
δ: death rate

as $P \nearrow$, death rate $kP \nearrow$ as well

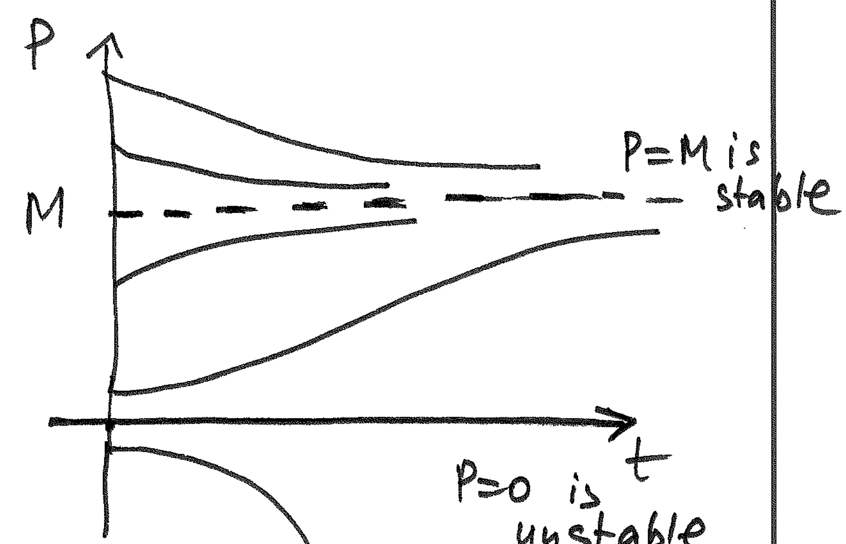
$\underbrace{\frac{dP}{dt}}_{P'} = k(M-P)P$: has 2 equilibrium solutions

$P=0$ & $P=M$

<u>phase plane</u>



$P' = kMP - kP^2$



$P=0$ is unstable equil. $sol^{\underline{n}}$

$P=M$ is stable equil. $sol^{\underline{n}}$

$$\frac{dP}{dt} = k(M-P)P$$

(a) $P \approx 0 \Rightarrow M-P \approx M \Rightarrow \dfrac{dP}{dt} \approx \underbrace{kM}_{>0} \cdot P$: exp growth eqⁿ

$\Rightarrow$ P grows exponentially when P is small

(b) $P \approx M \Rightarrow \dfrac{dP}{dt} \approx k(M-P)M$

or $\dfrac{dP}{dt} \approx kM(M-P)$: Newton's law of cooling/heating

Solution

$\dfrac{dP}{dt} = k(M-P)P$: separable ODE

$$\frac{dP}{(M-P)P} = k\,dt$$

$$\int \frac{dP}{(M-P)P} = \int k\,dt$$

Use <u>partial fraction decomposition</u>.

$$\frac{1}{\underbrace{(M-P)P}_{\xi}} = \frac{A}{M-P} + \frac{B}{P} = \frac{AP + B(M-P)}{\underbrace{(M-P)P}_{\xi}}$$

$$\Rightarrow 1 = AP + B(M-P)$$

$$1 = (A-B)P + M \cdot B$$

$$P^0: \quad 1 = M \cdot B \quad \Rightarrow \quad B = \frac{1}{M}$$

$$P = P^1: \quad 0 = A - B \quad \Rightarrow \quad A = B = \frac{1}{M}$$

$$\frac{1}{(M-P)P} = \frac{\frac{1}{M}}{M-P} + \frac{\frac{1}{M}}{P} = \frac{1}{M}\left(\frac{1}{M-P} + \frac{1}{P}\right)$$