Lecture 9

S2.1-2.2 **Population Models. Stability** (Cont'd)

Logistic Equation

$$\frac{dP}{dt} = k(M-P)P, \quad k>0, \; M>0$$

$$\frac{dP}{(M-P)P} = k\,dt$$

$$\frac{1}{(M-P)P} = \frac{1}{M}\left(\frac{1}{M-P} + \frac{1}{P}\right): \text{ partial fraction decomposition (see previous Lecture)}$$

$$\int \frac{dP}{(M-P)P} = \int k\,dt$$

$$\ln(a \cdot b) = \ln a + \ln b$$

$$\ln \frac{a}{b} = \ln a - \ln b$$

$$\frac{1}{M}\int \left(\frac{1}{M-P} + \frac{1}{P}\right) dP = \int k\,dt$$

$$\frac{1}{M}\left(-\ln|M-P| + \ln|P|\right) = kt + \tilde{C}$$

$$\frac{1}{M}\ln\left|\frac{P}{M-P}\right| = kt + \tilde{C} \qquad \Big| \cdot M$$

$$\ln\left|\frac{P}{M-P}\right| = kMt + \bar{C} \qquad \Big|\ \exp$$

$$\left|\frac{P}{M-P}\right| = e^{kMt+\bar{C}}$$

$$\frac{P}{M-P} = Ce^{kMt} \qquad \Big|\cdot(M-P) \qquad (1)$$

$$P = C(M-P)e^{kMt}$$

$$P = CMe^{kMt} - \underline{CP}e^{kMt} \Rightarrow P + CPe^{kMt} = CMe^{kMt}$$

$$P(1+Ce^{kMt}) = CMe^{kMt}$$

solve for P

$$\Rightarrow P(t) = \frac{CMe^{kMt}}{1+Ce^{kMt}} = \frac{M}{1+\frac{1}{C}e^{-kMt}}, \qquad k, M>0$$

$$\frac{CMe^{kMt}}{1+Ce^{kMt}} = \frac{M}{\frac{1}{C}e^{-kMt}+1}$$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{M}{1+\frac{1}{C}\underbrace{e^{-kMt}}_{\to 0}} = M:$$

$P=M$ is stable equil. solution

To find C, we use (1): $\dfrac{P}{M-P} = Ce^{kMt}$

$P(0) = P_0 \Rightarrow \dfrac{P_0}{M-P_0} = C$

$$\therefore \quad P(t) = \frac{M}{1 + \frac{M-P_0}{P_0} e^{-kMt}} = \boxed{\frac{MP_0}{P_0 + (M-P_0)e^{-kMt}} = P(t)}$$

Note

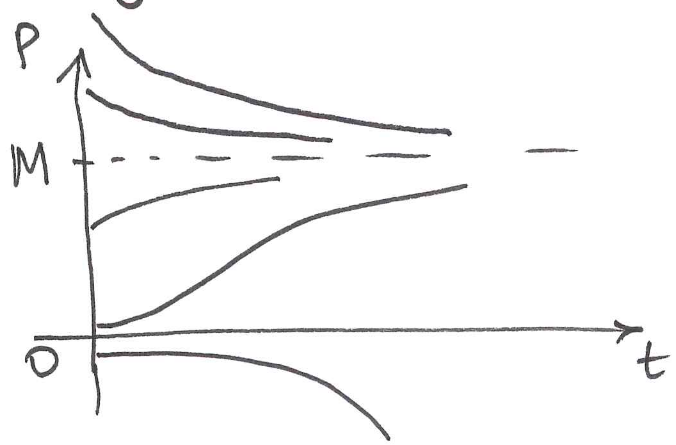
$$\left.\begin{array}{l} P_0 = 0 \Rightarrow P(t) = 0 \text{ for all } t \\ P_0 = M \Rightarrow P(t) = M \text{ for all } t \end{array}\right\} \quad P=0 \text{ \& } P=M \text{ are two equil. solutions}$$

Let $0 < P_0 < M$ or $P_0 > M \Rightarrow \lim_{t\to\infty} P(t) = M$

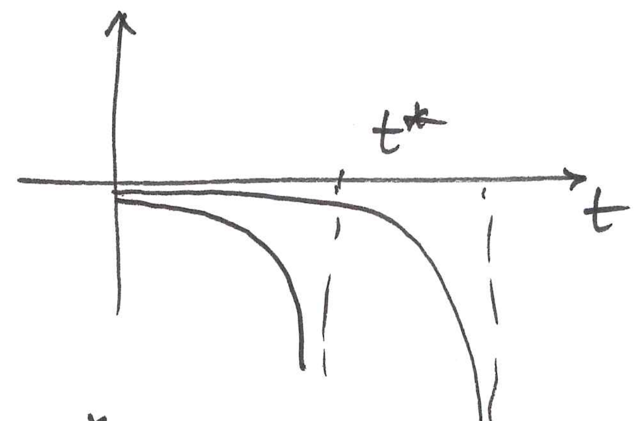
$\Rightarrow P(t) = 0$ is unstable equil. solution

$P(t) = M$ is stable — " —

This confirms what we already knew from stability analysis.



Solution curves



Q What happens when $P_0 < 0$?

$$P(t) = \frac{M P_0}{\underbrace{P_0}_{<0} + \underbrace{(M-P_0)}_{>0} \underbrace{e^{-kMt}}_{>0}}$$

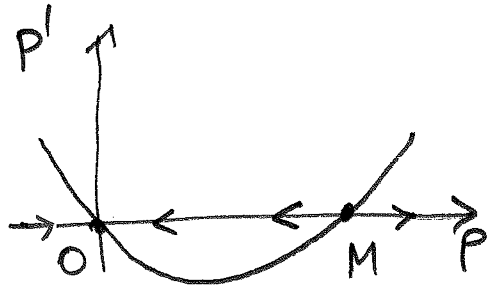
at some finite time t^*, $P_0 + (M-P_0)e^{-kMt^*} = 0$

$\Rightarrow \lim_{t \to t^*} P(t) = -\infty$

Doomsday / Extinction Model

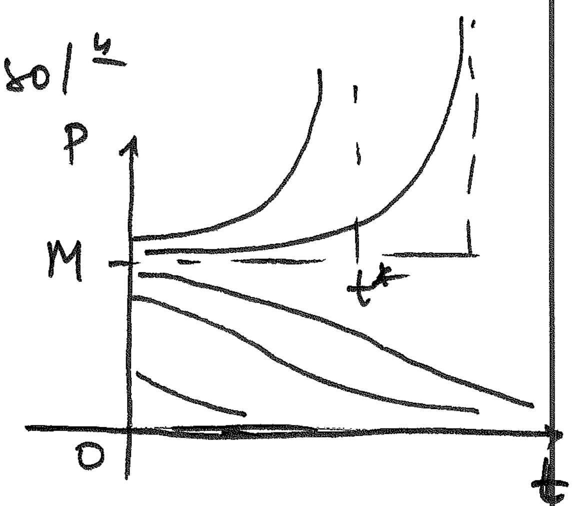
$$\frac{dP}{dt} = k P(P-M), \qquad k, M > 0$$

There are two equil. solutions: $P=0$, $P=M$



$P=0$: stable equil. sol$^{\underline{n}}$

$P=M$: unstable equil. sol$^{\underline{n}}$



Solution: $$P(t) = \frac{M P_0}{P_0 + (M-P_0)e^{kMt}}$$

if $0 < P_0 < M$, $\lim_{t\to\infty} P(t) = 0$: <u>extinction</u>

if $P_0 > M \Rightarrow$ there is some finite t^* at which

denominator $\underbrace{P_0}_{>0} + \underbrace{(M-P_0)}_{<0}\underbrace{e^{kMt^*}}_{>0} = 0$ and $\lim_{t\to t^*} P(t) = +\infty$: <u>doomsday</u>

Logistic Equation with Harvesting

$x = x(t)$

$\frac{dx}{dt} = ax - bx^2 - h$: logistic equation w/ harvesting

$a, b, h > 0$

or $\frac{dx}{dt} = \underbrace{kx(M-x) - h}_{\text{RHS does not depend on } t \text{ explicitly}}$ where $a = kM$, $b = k$

when $h = 0$, we get logistic eq$^{\underline{n}}$

$\Rightarrow$ this is an autonomous DE

Def a differential equation $\frac{dx}{dt} = f(x)$ is called autonomous.

does not depend on t explicitly

Consider

$$\frac{dx}{dt} = kx(M-x) - h$$

$$\frac{dx}{dt} = -k(x-H)(x-N)$$

if $(kM)^2 - 4kh > 0$

or $M^2 - \frac{4h}{k} > 0$

$$H, N = \frac{1}{2}\left(M \pm \sqrt{M^2 - 4\frac{h}{k}}\right)$$

$ax^2 + bx + c = a(x-x_1)(x-x_2)$

x_1, x_2 : roots $\qquad x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

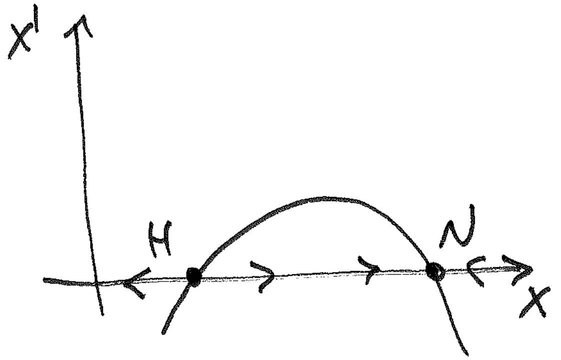
$-kx^2 + kMx - h = 0$

$$x_{1,2} = \frac{-kM \pm \sqrt{(kM)^2 - 4kh}}{-2k}$$

denote these roots by

H and N — w "+" sign

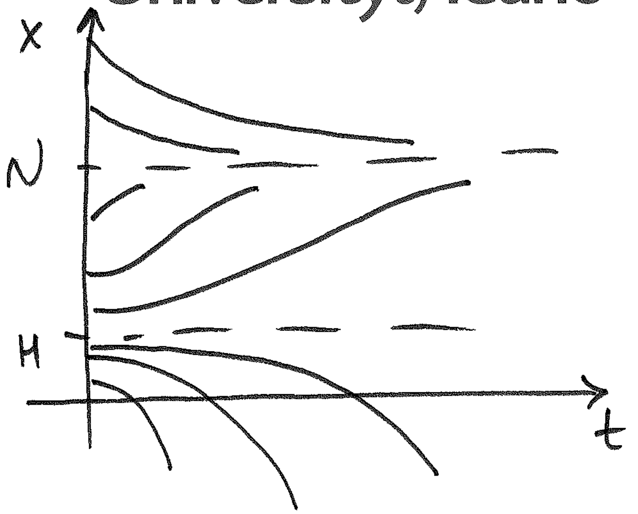
w/ "–" sign



H, N: 2 equil. sol^{ns}

$x = H$ is unstable equil. sol^{n}

$x = N$ is stable — " —



$x = N$: new limiting population size (due to harvesting)

Solution (by separation of variables and partial fraction decomposition):

$$x(t) = \frac{N(x_0 - H) - H(x_0 - N)e^{-k(N-H)t}}{(x_0 - H) - (x_0 - N)e^{-k(N-H)t}}$$

where $x(0) = x_0$.

If $x_0 < H$, then there is a finite time t_1:

$\lim_{t \to t_1} x(t) = -\infty$ but before t_1, $x(t)$ becomes zero: extinction

$x(t) = H$: threshold equilibrium solution that separates two different solution behaviour :

if $x_0 > H \Rightarrow \lim_{t\to\infty} x(t) = N$

if $x_0 < H \Rightarrow$ population becomes extinct in a finite time (due to harvesting)