3.5 Method of Undetermined Coefficients

Consider

$$P(D)y = R(x) \tag{3.1}$$

The general solution is $y = y_c + y_p$ where y_c is the complementary function (with arbitrary constants) and $P(D)y_c = 0$. y_p is the particular solution (with no arbitrary constants) and $P(D)y_p = R(x)$. Suppose that there is an operator (with constant coefficients) $A(D)$ called an *annihilator* such that $A(D)R(x) = 0$. If we operate on both sides of (3.1) with $A(D)$ we get a higher order equation

$$A(D)P(D)y = A(D)R(x) = 0$$

Consider this new higher order equation

$$A(D)P(D)y = 0 \tag{3.2}$$

To find the general solution of (3.2) we need the roots of the polynomial $P(D)A(D)$; they are $r_1, r_2, \ldots, r_j,\ q_1, q_2, \ldots, q_k$, where $r_1, r_2, \ldots, r_j$ are roots of $P(D)$ and $q_1, q_2, \ldots, q_k$ are the roots of $A(D)$. Thus the general solution of (3.2) is

$$y = y_c + y_q$$

where y_c is generated by the roots of $P(D)$ and y_q is generated by the roots of $A(D)$.

Note 1. $r_1, r_2, \ldots, r_j,\ \ q_1, q_2, \ldots, q_k$ are roots of a (single) polynomial $P(D)A(D)$, thus make sure that if one of the q's is a repeated root to treat it properly.

Note 2. The general solution of (3.1) is also "a" solution of (3.2)

$$A(D)P(D)[y_c + y_p] = A(D)R(x) = 0$$

Thus, since $y_c + y_p$ is "a" solution of (3.2) and $y_c + y_q$ is "the general solution" of (3.2), $y_c + y_p$ must be contained in $y_c + y_q$, i.e.

$$(y_c + y_p) \subset (y_c + y_q) \quad \text{or} \quad y_p \subset y_q$$

We call y_q the "candidate" for the particular solutions and use the method of undetermined coefficients to evaluate the constants in y_q and thus find y_p.

EXAMPLE Find the candidate for y_p, the particular solution.

$$y'' - 3y' + 2y = 8\cos 2x + 6\,e^{4x}$$

$$(D^2 - 3D + 2)y = 8\cos 2x + 6\,e^{4x} \qquad A(D) = (D^2+4)(D-4)$$

the higher order DE is

$$\overset{P(D)}{[(D-1)(D-2)]}\,\overset{A(D)}{[(D^2+4)(D-4)]}\,y = 0$$

$$1, \qquad 2; \qquad \pm 2i, \qquad 4$$

$$y = \underbrace{C_1\,e^{x} + C_2\,e^{2x}}_{y_c} + \underbrace{K_1\cos 2x + K_2\sin 2x + K_3\,e^{4x}}_{y_q}$$

EXAMPLE

$$(D^2+1)(D-1)(D+4)y = 10\cos 4x + 6x\,e^{x} - 12\,e^{-4x},$$

$$A(D) = (D^2+16)(D-1)^2(D+4)$$

the higher order DE is

$$[(D^2+1)(D-1)(D+4)][(D^2+16)(D-1)^2(D+4)]y = 0$$

$$\pm i, \qquad 1, \qquad -4; \qquad \pm 4i, \qquad 1, 1, \qquad -4$$

$$y = \underbrace{C_1\cos x + C_2\sin x + C_3\,e^{x} + C_4\,e^{-4x}}_{y_c}$$

$$+ \underbrace{K_1\cos 4x + K_2\sin 4x + K_3 x\,e^{x} + K_4 x^2\,e^{x} + K_5 x\,e^{-4x}}_{y_q}$$

EXAMPLE

$$D^3(D-1)^2(D^2+1)y = 4x - 7x^2\,\mathrm{e}^x + 9x^2\,\mathrm{e}^{-x} + 3\cos x$$

$$A(D) = D^2(D-1)^3(D+1)^3(D^2+1)$$

the higher order DE is

$$[D^3(D-1)^2(D^2+1)][D^2(D-1)^3(D+1)^3(D^2+1)]y = 0$$

$$0,0,0,\ 1,1 \qquad \pm\, i; \quad 0,0,\ 1,1,1,\ -1,-1,-1,\ \pm i$$

$$y = \underbrace{C_1 + C_2x + C_3x^2 + C_4\,\mathrm{e}^x + C_5x\,\mathrm{e}^x + C_6\cos x + C_7\sin x}_{y_c}$$

$$\underbrace{\begin{aligned}&+K_1x^3 + K_2x^4 + K_3x^2\,\mathrm{e}^x + K_3x^2\,\mathrm{e}^x + K_5x^4\,\mathrm{e}^x + K_6\,\mathrm{e}^{-x}\\ &+K_7x\,\mathrm{e}^{-x} + K_8x^2\,\mathrm{e}^{-x} + K_9x\cos x + K_{10}x\sin x\end{aligned}}_{y_q}$$

What do we do with y_q the candidate for y_p? ANS Substitute into the DE and evaluate the K's to obtain y_p.

`EXAMPLE` Solve

$$y'' - 3y' + 2y = x\,e^{2x} + \sin x \quad \text{with} \quad y(0) = 1.3 \quad y'(0) = 4.1$$

$$(D^2 - 3D + 2)y = x\,e^{2x} + \sin x \qquad A(D) = (D-2)^2(D^2+1)$$

the higher order DE is

$$[(D-1)(D-2)][(D-2)^2(D^2+1)]y = 0$$

$$1, \qquad 2; \qquad 2, 2, \qquad \pm i$$

$$y = \underbrace{C_1\,e^{x} + C_2\,e^{2x}}_{y_c} + \underbrace{K_1 x\,e^{x} + K_2 x^2\,e^{2x} + K_3\cos x + K_4 \sin x}_{y_q}$$

TO FIND y_p we operate on y_q with $P(D)$ and set it equal to the functions on the right hand side.

$$(D^2 - 3D + 2)[K_1 x\,e^{x} + K_2 x^2\,e^{2x} + K_3\cos x + K_4\sin x] = x\,e^{2x} + \sin x$$

$$\begin{array}{|c|} \hline +2 \\ \hline \end{array} \quad y_q = K_2x^2\,e^{2x} + K_1 x\,e^{2x} + K_3\cos x + K_4\sin x$$

$$\begin{array}{|c|} \hline -3 \\ \hline \end{array} \quad y_q' = 2K_2x^2\,e^{2x} + (2K_2 + 2K_1)x\,e^{2x} + K_1\,e^{2x} + K_4\cos x - K_3\sin x$$

$$\begin{array}{|c|} \hline +1 \\ \hline \end{array} \quad y_q'' = 4K_2x^2\,e^{2x} + (8K_2 + 4K_1)x\,e^{2x} + (2K_2 + 4K_1)\,e^{2x} - K_3\cos x - K_4\sin x$$

$$2K_2 x\,e^{2x} + (2K_2 + K_1)\,e^{2x} + (K_3 - 3K_4)\cos x + (3K_3 + K_4)\sin x = x\,e^{2x} + \sin x$$

We now equate the coefficients of $x\,e^{2x}$, e^{2x}, $\cos x$, $\sin x$ on the left side with those on the right. Can we do this? YES. Why? The set $\{x\,e^{2x}, e^{2x}, \cos x, \sin x\}$ is linearly independent.

$$2K_2 = 1 \qquad 2K_2 + K_1 = 0 \qquad K_3 - 3K_4 = 0 \qquad 3K_3 + K_4 = 1$$

$$\boxed{K_2 = \frac{1}{2}} \qquad \boxed{K_1 = -1} \qquad \boxed{K_3 = .3} \qquad \boxed{K_4 = .1}$$

General solution

$$y = \underbrace{C_1\,e^{x} + C_2\,e^{2x}}_{\text{complementary function}} + \underbrace{\frac{1}{2}x^2\,e^{2x} - x\,e^{2x} + .1\sin x + .3\cos x}_{\text{particular solution}}$$

$$y' = C_1\,e^{x} + 2C_2\,e^{2x} + x\,e^{2x} + x^2\,e^{2x} - 2x\,e^{2x} - e^{2x} + .1\cos x - .3\sin x$$

From initial conditions

$$\begin{array}{l} 1.3 = C_1 + C_2 + .3 \\ 4.1 = C_1 + 2C_2 + .1 - 1 \end{array} \Rightarrow \begin{array}{l} C_1 = -3 \\ C_2 = 4 \end{array}$$

$$\boxed{y = -3\,e^{x} + 4\,e^{2x} + \frac{1}{2}x^2\,e^{2x} - x\,e^{2x} + .1\sin x + .3\cos x}$$