

12/14/2014

Math 310: Final Review#21
S.4

$$2y \frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}}, \quad y(5)=2$$

$$\frac{dy}{dx} = \frac{x}{2y\sqrt{x^2-16}} = \underbrace{\frac{1}{y}}_{f^2 \text{ of } y} \cdot \underbrace{\frac{x}{2\sqrt{x^2-16}}}_{f^2 \text{ of } x} \quad \begin{array}{l} \text{: separable} \\ \text{ODE} \\ \text{(1st order} \\ \text{nonlinear)} \end{array}$$

$$y \, dy = \frac{x \, dx}{2\sqrt{x^2-16}}$$

$$\int y \, dy = \int \frac{x \, dx}{2\sqrt{x^2-16}}$$

$$\frac{y^2}{2} = \frac{1}{2} \sqrt{x^2-16} + C \quad / \cdot 2$$

$$\int \frac{x \, dx}{2\sqrt{x^2-16}} = \left| \begin{array}{l} u = x^2-16 \\ du = 2x \, dx \end{array} \right| = \int \frac{du}{2 \cdot 2 \sqrt{u}} = \frac{1}{2} \sqrt{u} + C$$

$$= \frac{1}{2} \sqrt{x^2-16} + C$$

$$y^2 = \sqrt{x^2-16} + \tilde{C}$$

$$y(5)=2 \Rightarrow 2^2 = \sqrt{5^2-16} + \tilde{C}$$

$$4 = 3 + \tilde{C} \Rightarrow \tilde{C} = 1$$

$$\Rightarrow y^2 = \sqrt{x^2 - 16} + 1$$

$$y(5) = 2$$

$$y = \pm \sqrt{\sqrt{x^2 - 16} + 1}$$

choose '+' solution

$$\therefore \boxed{y = \sqrt{\sqrt{x^2 - 16} + 1}}$$

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S1.5

$$xy' - y = x, \quad y(1) = 7$$

1st order linear DE

Multiply both sides

of DE by $\frac{1}{x}$.

$$y' + P(x)y = Q(x)$$

$$p(x) = e^{\int P(x) dx} \quad : \text{integrating factor}$$

$$y' - \underbrace{\frac{1}{x}}_P y = \underbrace{1}_Q$$

$$py = \int pQ dx + C$$

$$p(x) = e^{\int P(x) dx} = e^{\int (-\frac{1}{x}) dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

$$py = \int pQ dx + C$$

$$e^{\ln a} = a$$

$$\frac{1}{x} \cdot y = \int \frac{1}{x} \cdot 1 dx + C$$

$$a \ln b = \ln b^a$$

$$\frac{1}{x} y = \ln x + C$$

$$\text{IC: } y(1) = 7 \Rightarrow \frac{1}{1} \cdot 7 = \ln 1 + C \Rightarrow \boxed{C = 7}$$

$$\frac{1}{x} y = \ln x + 7 \Rightarrow \boxed{y = x(\ln x + 7)}$$

Method of undetermined coefficients

#48

S 9.5

$$y'' - 2y' - 8y = 3e^{-2x} \quad (1)$$

$$(D^2 - 2D - 8)y = 3e^{-2x}$$

$$-2, 4 \quad -2 \quad A(D) = D + 2$$

$$(D + 2)(D - 4)y = 3e^{-2x}$$

Higher order DE is

$$[(D + 2)(D - 4)](D + 2)y = 0$$

$$-2, 4; -2$$

$$y(x) = C_1 e^{-2x} + C_2 e^{4x} + K_1 x e^{-2x}$$

$y_g = K_1 x e^{-2x}$: candidate for particular solution

Substitute y_g into (1) to find K_1

$$(-8) \quad y_g = K_1 x e^{-2x}$$

$$(-2) \quad y_g' = K_1 e^{-2x} + K_1 x (-2) e^{-2x} = K_1 e^{-2x} - 2K_1 x e^{-2x}$$

$$y_g'' = -2K_1 e^{-2x} - 2K_1 e^{-2x} + 4K_1 x e^{-2x}$$

$$\textcircled{1} \quad \underline{y'' = -4K_1 e^{-2x} + 4K_1 x e^{-2x}}$$

$$(-8K_1 - 2(-2)K_1 + 4K_1) x e^{-2x} + (-2K_1 - 4K_1) e^{-2x} = 3e^{-2x}$$

$\{e^{-2x}, x e^{-2x}\}$: linear independent

$$x e^{-2x}: \quad -8K_1 + 4K_1 + 4K_1 = 0 \quad -8K_1 + 8K_1 = 0 \quad \checkmark$$

$$e^{-2x}: \quad -6K_1 = 3 \quad \Rightarrow \quad \boxed{K_1 = -\frac{1}{2}}$$

$$\therefore \boxed{y_p(x) = -\frac{1}{2} x e^{-2x}}$$

General solution is

$$\boxed{y(x) = C_1 e^{-2x} + C_2 e^{4x} - \frac{1}{2} x e^{-2x}}$$

Variation of parameters

#48

S3.5

$$y'' - 2y' - 8y = 3e^{-2x}$$

$y'' - 2y' - 8y = 0$: associated homogeneous eqⁿ

$$(D^2 - 2D - 8)y = 0$$

$$(D+2)(D-4)y = 0$$

-2, 4

$$\Rightarrow \boxed{y_c = C_1 e^{-2x} + C_2 e^{4x}}$$

Assume

$$y_p(x) = A_1(x)e^{-2x} + A_2(x)e^{4x}$$

To find A_1, A_2 , we solve for A_1', A_2' the following system of equations

$$\begin{cases} y_1 A_1' + y_2 A_2' = 0 \\ y_1' A_1' + y_2' A_2' = \frac{R(x)}{a_2(x)} \end{cases}$$

where y_1, y_2 are linearly independent solutions of the associated homogeneous DE, i.e.

$$y_1 = e^{-2x}, \quad y_2 = e^{4x}$$

$$R(x) = 3e^{-2x}, \quad a_2 = 1$$

$$\begin{pmatrix} e^{-2x} & e^{4x} \\ -2e^{-2x} & 4e^{4x} \end{pmatrix} \begin{pmatrix} A_1' \\ A_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 3e^{-2x}/1 \end{pmatrix}$$

Solve using Cramer's rule

$$\begin{aligned} \Delta &= \begin{vmatrix} e^{-2x} & e^{4x} \\ -2e^{-2x} & 4e^{4x} \end{vmatrix} = 4e^{-2x}e^{4x} + 2e^{-2x}e^{4x} \\ &= 4e^{2x} + 2e^{2x} = 6e^{2x} \end{aligned}$$

$$A_1 = \begin{vmatrix} 0 & e^{4x} \\ 3e^{-2x} & 4e^{4x} \end{vmatrix} = -3e^{2x}$$

$$A_2 = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & 3e^{-2x} \end{vmatrix} = 3e^{-4x}$$

Then

$$A_1' = \frac{\Delta_1}{\Delta} = \frac{-3e^{2x}}{6e^{2x}} = -\frac{1}{2}$$

$$A_1(x) = \int \left(-\frac{1}{2}\right) dx = \boxed{-\frac{1}{2}x = A_1(x)}$$

$$A_2' = \frac{\Delta_2}{\Delta} = \frac{3e^{-4x}}{6e^{2x}} = \frac{1}{2}e^{-6x}$$

$$A_2(x) = \int \frac{1}{2}e^{-6x} dx = \boxed{-\frac{1}{12}e^{-6x} = A_2(x)}$$

Hence,

$$\begin{aligned} y_p(x) &= A_1 \cdot y_1 + A_2 \cdot y_2 = -\frac{1}{2}x \cdot e^{-2x} - \frac{1}{12}e^{-6x} \cdot e^{4x} \\ &= -\frac{1}{2}x e^{-2x} - \frac{1}{12}e^{-2x} \end{aligned}$$

Then, the general solution is

$$y(x) = \underbrace{C_1 e^{-2x} + C_2 e^{4x}}_{y_e} - \frac{1}{2}x e^{-2x} - \frac{1}{12}e^{-2x} \quad \textcircled{=}$$

y_p

Find inverse Laplace transform.

#13
S7.4

$$F(s) = \frac{s}{(s-3)(s^2+1)} = \frac{A}{s-3} + \frac{Bs+C}{s^2+1} =$$

$$= \frac{A(s^2+1) + (Bs+C)(s-3)}{(s-3)(s^2+1)}$$

$$\Rightarrow s = A(s^2+1) + (Bs+C)(s-3)$$

$$s^2: 0 = A + B \quad \Rightarrow B = -A$$

$$s^1: 1 = -3B + C$$

$$s^0: 0 = A - 3C \quad C = \frac{A}{3}$$

$$1 = -3(-A) + \frac{A}{3}$$

$$1 = \left(3 + \frac{1}{3}\right)A \Rightarrow 1 = \frac{10}{3}A \Rightarrow A = \frac{3}{10} = 0.3$$

$$B = -A = -0.3, \quad C = \frac{A}{3} = 0.1$$

$$\therefore F(s) = \frac{0.3}{s-3} + \frac{-0.3s+0.1}{s^2+1} =$$

$$= 0.3 \frac{1}{s-3} - 0.3 \frac{s}{s^2+1} + 0.1 \frac{1}{s^2+1}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 0.3e^{3t} - 0.3 \cos t + 0.1 \sin t$$

$$\textcircled{=} \tilde{C}_1 e^{-2x} + C_2 e^{4x} - \frac{1}{2} x e^{-2x} \quad \text{---yp}$$

$$\tilde{C}_1 = C_1 - \frac{1}{2} \quad \text{: arbitrary constant}$$

\text{arbitrary constant}

#21

S7.4

$$f(t) = \frac{e^{3t} - 1}{t}$$

Thm: integration of transforms

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma) d\sigma \quad |f(t)| \leq M e^{at} \text{ as } t \rightarrow \infty$$

$$\mathcal{L}\left\{\frac{e^{3t} - 1}{t}\right\} = \int_s^\infty \mathcal{L}\{e^{3t} - 1\} d\sigma \textcircled{=}$$

$$\mathcal{L}\{e^{3t} - 1\} = \frac{1}{s-3} - \frac{1}{s}$$

$$\textcircled{=} \int_s^\infty \left(\frac{1}{\sigma-3} - \frac{1}{\sigma}\right) d\sigma = \left(\ln|\sigma-3| - \ln|\sigma|\right) \Big|_s^\infty =$$

$$= \ln\left|\frac{\sigma-3}{\sigma}\right| \Big|_s^\infty = \cancel{\ln 1} - \ln\left(\frac{s-3}{s}\right) = \boxed{\ln \frac{s}{s-3}} \quad s > 3$$

$$\lim_{\sigma \rightarrow \infty} \frac{\sigma-3}{\sigma} = \frac{1}{1} = 1 \Rightarrow \ln 1 = 0$$

$$F(s) = \frac{s}{(s-3)(s^2+1)} = \frac{1}{s-3} \cdot \frac{s}{s^2+1} = \mathcal{L}\{e^{3t}\} \cdot \mathcal{L}\{\cos t\}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{3t} * \cos t =$$

$$= \int_0^t e^{3\tau} \cos(t-\tau) d\tau = \int_0^t \underbrace{e^{3(t-\tau)}}_{e^{3t} e^{-3\tau}} \cos \tau d\tau =$$

$$= e^{3t} \underbrace{\int_0^t e^{-3\tau} \cos \tau d\tau}_I : \text{twice by parts to get equation for } I$$

Thus: $f * g = g * f$