

1/9/2013

Introduction

linear optimization or linear programming is a subfield of optimization. Optimization is a subfield of applied mathematics.

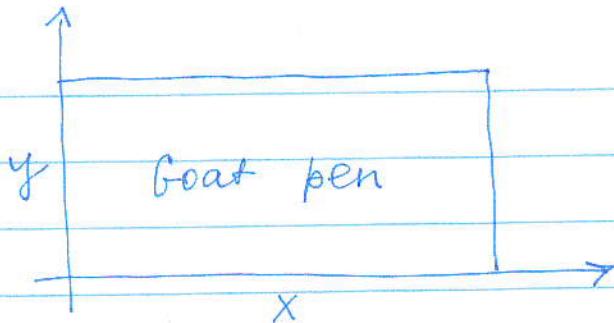
In applied mathematics, mathematical methods are used to study real-world problems.

Optimization, roughly speaking, is about making things better. Examples including helping a factory to maximize profits or to minimize waste, or to develop proper and sue most nutritious food for animals. In all these examples, one has to optimize some function to make things better, through some decisions.

Ex A simple problem from calculus.

Goats are used to eat grass where there are lots of rocks and hills.

Suppose we need to construct a goat pen to keep some goats. We are given 100m of fencing. We want to construct a rectangular pen with the largest possible area.



We know that

$$(1) \quad 2x + 2y = 100 \quad | :2$$

The area of this pen is $A(x,y) = xy \rightarrow \max$

$$\text{From (1)} \Rightarrow x + y = 50 \Rightarrow y = 50 - x$$

$$\therefore A(x,y) = xy \Rightarrow A(x) = x(50-x) = 50x - x^2$$

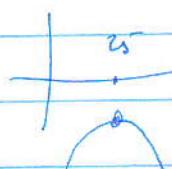
~~$\partial A / \partial x = 50 - 2x = 0 \Rightarrow x = 25$~~

$$\begin{aligned} \frac{dA}{dx} &= 50 - 2x = 2(25 - x) = 0 \Rightarrow x = 25 \\ \Rightarrow y &= 50 - x = 25 \end{aligned}$$

To check that $A(x)$ has a local maximum at $x = 25$, we check

$$\left. \frac{d^2A}{dx^2} \right|_{x=25} = -2 < 0 \Rightarrow A(x) \text{ has local max at } x = 25$$

Another way to see this is to note that $A(x) = 50x - x^2$ is "upside-down" parabola



Hence, the square pen with $x=y=25$ will have the largest area.

Exercise 1 A canning company is producing canned corn for the holidays. They know that each family prefers their corn in units of 12 fluid ounces. Assuming that metal costs 1 cent per square inch and 1 fluid ounce is about 1.8 cubic inches, compute the ideal height and radius of a can of corn assuming that cost is to be minimized.

Hints



$$\text{Surface area: } 2\pi rh + 2\pi r^2 \rightarrow \min$$

$$\text{Volume: } \pi r^2 h$$

The cost is related to the surface area, i.e. since metal is priced per square inch you want to minimize surface area with condition/constraint of fixed volume.

A general maximization problem

We will use the goat pen example to introduce necessary terminology.

The area function can be written as a function that maps \mathbb{R}^2 to \mathbb{R} , i.e.

$$A: \mathbb{R}^2 \rightarrow \mathbb{R}$$

\mathbb{R}^2 is a domain of A , \mathbb{R} is a range.

The objective of the problem was to maximize the area by choosing values x and y . In optimization theory, if we want to maximize or minimize a function, we call it an objective function. In general, an objective function is a mapping

$$z: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$$

Here D is a domain of z .

Recall

Def Let $z: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$. The point $\vec{x}^* \in D$ is a global maximum of z if

$$z(\vec{x}^*) \geq z(\vec{x}) \text{ for all } \vec{x} \in D$$

point of

A point $\vec{x}^* \in D$ is a local maximum of z if there exists a neighborhood $S \subseteq D$ such that

$$z(\vec{x}^*) \geq z(\vec{x}) \text{ for all } \vec{x} \in S$$

Note Global and local points of minimum can be defined in a similar way.

In the goat pen example, we constrained our choice of x and y by the fact that $2x + 2y = 100$. This is called a constraint equality.

If we did not have to use all 100m of fencing, we would have $2x + 2y \leq 100$: inequality constraint. In optimization problems, there are usually many constraints. The set of all points in \mathbb{R}^n that satisfy the ~~constraints~~ constraints form a feasible set (or feasible region). Our problem is to decide about the best values of x and y that will maximize the area $A(x, y)$. Therefore, the variables x and y are called decision variables.

Let $z: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$. The general maximization problem can be written as

$$\begin{aligned} & \text{maximize} && z(x_1, \dots, x_n) \\ & \text{subject to} && g_1(x_1, \dots, x_n) \leq b_1 \\ & (\text{or such that}) && \quad \quad \quad \cdots \\ & && g_m(x_1, \dots, x_n) \leq b_m \end{aligned} \quad \left\{ \begin{array}{l} \text{inequality} \\ \text{constraints} \end{array} \right.$$

$$(**) \quad \left\{ \begin{array}{l} h_1(x_1, \dots, x_n) = r_1 \\ \vdots \\ h_l(x_1, \dots, x_n) = r_l \end{array} \right. \quad \left. \begin{array}{l} \text{equality} \\ \text{constraints} \end{array} \right.$$

This is called mathematical programming
problem

1/11/2013

We can re-write the goat pen problem using general formulation of a maximization problem:

$$\begin{array}{ll} \text{maximize} & A(x, y) = xy \\ \text{subject} & 2x + 2y = 100 \\ \text{to} & x \geq 0 \\ & y \geq 0 \end{array} \quad \left. \right\} \quad (1)$$

Conditions $x \geq 0, y \geq 0$: distances cannot be negative

We can also write $g_1(x, y) = -x \leq 0$ and $g_2(x, y) = -y \leq 0$.

Now problem (1) is written in the form (xtx) (see lecture 1/9/2013 at the end).

Note We can write inequality constraint as equality constraint.

$$tx = \sum_{j=1}^n a_{ij} x_j \geq b_i$$



$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

a_{21} is in the 2nd row
and 1st column

$$\sum_{j=1}^n a_{ij} x_j = a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n \geq b_i$$



We can introduce an additional variable x_{n+i} , called surplus variable, and write

$$\sum_{j=1}^n a_{ij}x_j - x_{n+i} = b_i, \quad x_{n+i} \geq 0$$

\oplus

We have formulated the general maximization problem. We can formulate the general minimization problem by replacing max to min and reversing inequality signs.

An alternative way to solve a minimization problem is to consider a maximization problem for $-z(x_1, x_2, \dots, x_n)$.

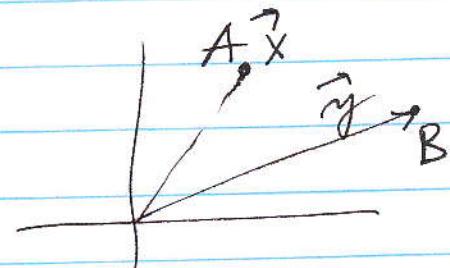
Some geometry for Optimization

Partial derivatives review

$$\underline{\text{ex}} \quad z(x, y) = xy^2$$

$$\frac{\partial z}{\partial x} = y^2 \quad \frac{\partial z}{\partial y} = 2xy$$

$$\vec{x} = (x_1, \dots, x_n), \quad \vec{y} = (y_1, \dots, y_n)$$



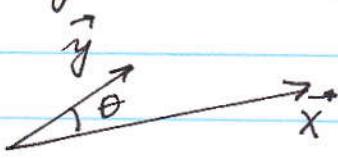
$$\text{dist}(A, B) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \dots + (y_n - x_n)^2}$$

Let $\vec{x} \in \mathbb{R}^n$: n -dimensional vector
 $\vec{x} = (x_1, \dots, x_n)$

Def For $\vec{x}, \vec{y} \in \mathbb{R}^n$, the dot product or scalar product is

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^n x_i \cdot y_i = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

There is an alternative def of the dot product. Let θ be the angle between \vec{x} and \vec{y} .



$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \cdot \|\vec{y}\| \cdot \cos \theta$$

$\backslash \quad /$
length of \vec{x}, \vec{y}

This comes from the law of cosines.

Claim The following is satisfied:

- (1) The angle θ between \vec{x} and \vec{y} is less than $\frac{\pi}{2}$ (θ is acute) iff $\vec{x} \cdot \vec{y} > 0$
- (2) The angle θ between \vec{x} and \vec{y} is exactly $\frac{\pi}{2}$ (i.e. \vec{x} and \vec{y} are orthogonal) $\Leftrightarrow \vec{x} \cdot \vec{y} = 0$
- (3) The angle θ is greater than $\frac{\pi}{2}$ (θ is obtuse) iff $\vec{x} \cdot \vec{y} < 0$.

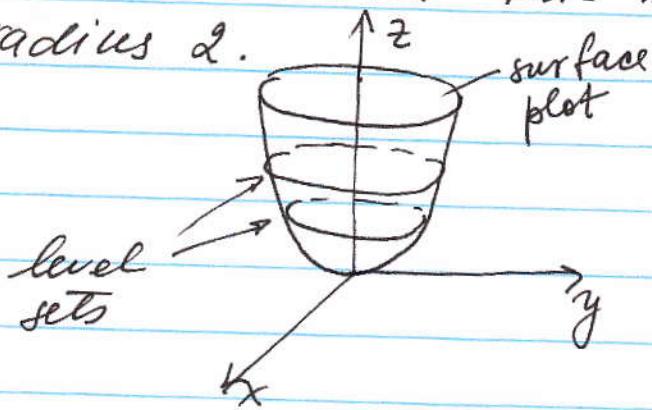
Def Let $z: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be a function, then the graph of z is the set of $n+1$ tuples:

$$\left\{ (\vec{x}, z(\vec{x})) : \vec{x} \in D \right\} \subset \mathbb{R}^{n+1}$$

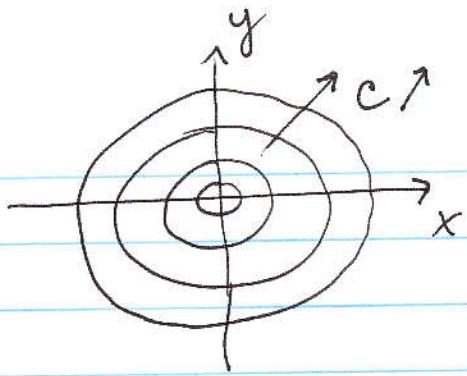
Example When $z: D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ ($y = z(x)$ a function of one variable), then the graph of z is what we know: it is a set of all possible pairs $(x, y) \in \mathbb{R}^2$ such that $y = z(x)$.

Def Let z be a function and $c \in \mathbb{R}$. Then the level set of value c for function z is the set of all $\vec{x} \in D$ such that $z(\vec{x}) = c$.

Ex Consider $z = x^2 + y^2$. The level set at value 4 of function z is the set of all pairs (x, y) such that $x^2 + y^2 = 4$. We can see that this is the circle with radius 2.



elliptic paraboloid
(because of elliptic
and parabolic
traces)



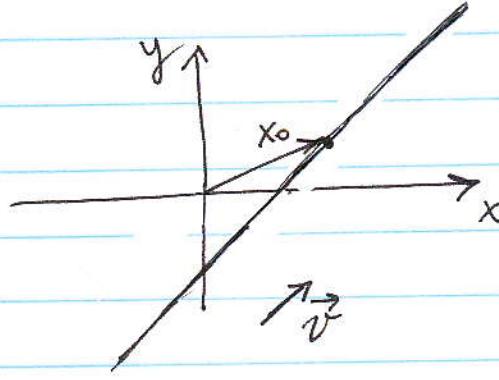
Contour plot

radii increase as $c \uparrow$

Def Let $\vec{x}_0, \vec{v} \in \mathbb{R}^n$. Then the line defined by vectors \vec{x}_0 and \vec{v} is the function

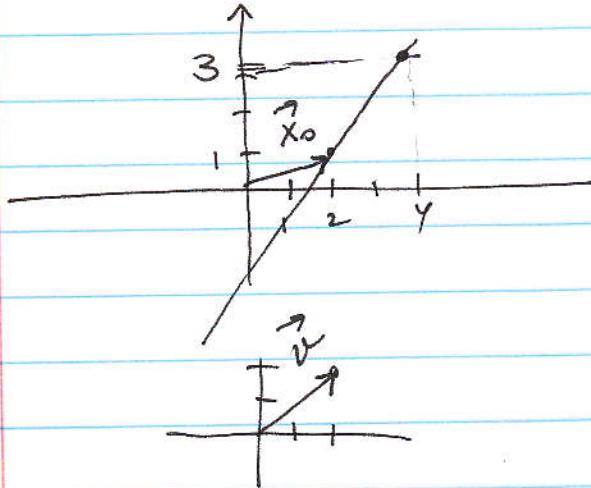
$$\vec{l}(t) = \vec{x}_0 + t \cdot \vec{v}$$

$$-\infty < t < \infty$$



Vector \vec{v} : direction of line \vec{l}

Ex Let $\vec{x}_0 = (2, 1)$ and $\vec{v} = (2, 2)$



$$\vec{l} = \vec{x}_0 + t \cdot \vec{v}$$

$$(x, y) = (2, 1) + t(2, 2)$$

$$\begin{aligned} x &= 2 + 2t \\ y &= 1 + 2t \end{aligned} \quad \left. \begin{array}{l} \text{parametric} \\ \text{equation of} \\ \text{the line} \end{array} \right.$$

$$t=1 \Rightarrow x = 2+2=4$$

$$y = 1+2=3$$

1/14/2013

Directional Derivatives and Gradients

Def

Let $z: \mathbb{R}^n \rightarrow \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n$ some vector
Then the directional derivative of z at $\vec{x}_0 \in \mathbb{R}^n$ in the direction of vector \vec{v} is

$$\left. \frac{d}{dt} z(\vec{x}_0 + t\vec{v}) \right|_{t=0}$$

Claim Directional derivative of z at \vec{x}_0 in
direction of \vec{v} is

$$\left. \frac{d}{dt} z(\vec{x}_0 + t\vec{v}) \right|_{t=0} = \lim_{h \rightarrow 0} \frac{z(\vec{x}_0 + h\vec{v}) - z(\vec{x}_0)}{h}$$

aside

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Def Let $z: \mathbb{R}^n \rightarrow \mathbb{R}$ be a function and $\vec{x}_0 \in \mathbb{R}^n$.
Then the gradient of z at \vec{x}_0 is a vector in \mathbb{R}^n
given by:

$$\nabla z(\vec{x}_0) = \left(\frac{\partial z}{\partial x_1}(\vec{x}_0), \frac{\partial z}{\partial x_2}(\vec{x}_0), \dots, \frac{\partial z}{\partial x_n}(\vec{x}_0) \right)$$

Gradients are very important in optimization.
One of their properties is relationship between
gradients and directional derivatives.

Claim If $z: \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable,
then the directional derivative of z
at \vec{x}_0 in the direction of \vec{v} is

$$\frac{d}{dt} z(\vec{x}_0 + t\vec{v}) \Big|_{t=0} = \nabla z(\vec{x}_0) \cdot \vec{v}$$

↑
dot product

Two important properties of gradients:

- (i) they always point in the direction of steepest ascent of a function with respect to the level curves of a function
- (ii) they are perpendicular (orthogonal) to the level curves of a function normal

Claim Let $z: \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable.

If $\nabla z(\vec{x}_0) \neq 0$, then $\nabla z(\vec{x}_0)$ points in the direction in which z increases fastest.

Indeed,

$$\frac{d}{dt} z(\vec{x}_0 + t\vec{v}) \Big|_{t=0} = \nabla z(\vec{x}_0) \cdot \vec{v} \quad \text{≡}$$

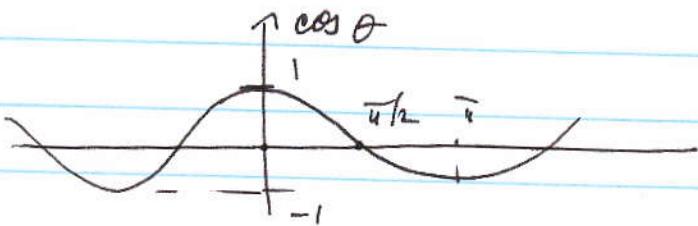
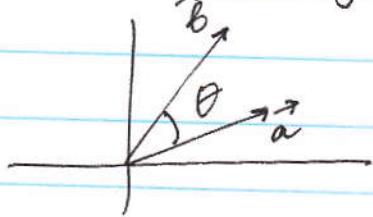
directional

deriv. of z at \vec{x}_0
in dir. of \vec{v}

let \vec{v} s.t. $\|\vec{v}\|=1$

$$\text{≡} \|\nabla z(\vec{x}_0)\| \cdot \|\vec{v}\| \cdot \cos \theta, \text{ where } \|\cdot\|: \text{length of vector}$$

θ is the angle between vectors, $0 \leq \theta \leq \pi$



$\cos \theta$ is the largest when $\cos \theta = 1 \Rightarrow \theta = 0$
 \Rightarrow vectors $\nabla z(\vec{x}_0)$ and \vec{v} are parallel
 \Rightarrow direct. derivative is largest when direction $\vec{v} \parallel \nabla z(\vec{x}_0)$.

Claim Let $z: \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable and let \vec{x}_0 lie in the level set S defined by $z(\vec{x}) = k$ for fixed $k \in \mathbb{R}$. Then $\nabla z(\vec{x}_0)$ is normal to the set S in the sense that if \vec{v} is a tangent vector at $t=0$ of a path $\vec{c}(t)$ completely contained in S with $\vec{c}(0) = \vec{x}_0$, then $\nabla z(\vec{x}_0) \cdot \vec{v} = 0$

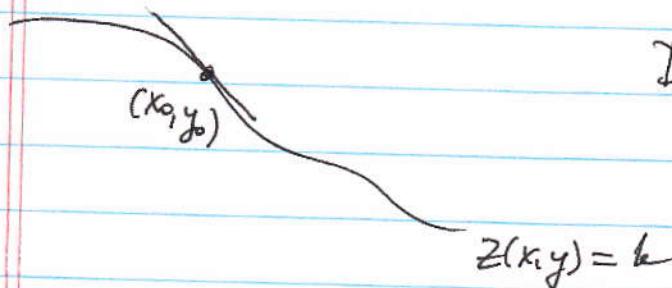
Ex $z(x, y) = x^4 + y^2 + 2xy$ and $\vec{x}_0 = (1, 1)$

$$\nabla z = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right) = (4x^3 + 2y, 2y + 2x)$$

$$\nabla z \Big|_{(1,1)} = (6, 4)$$

Consider a partial case: $z: \mathbb{R}^2 \rightarrow \mathbb{R}$, i.e. $z = z(x, y)$
 Level set is defined implicitly by a level curve

$$(1) \quad z(x, y) = k, \quad k \text{ is real}$$



Differentiate (1) implicitly
 wrt x to find $\frac{dy}{dx}$

$$\frac{d}{dx} \Big|_{z(x, y) = k} \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}$$

slope of
 tangent line

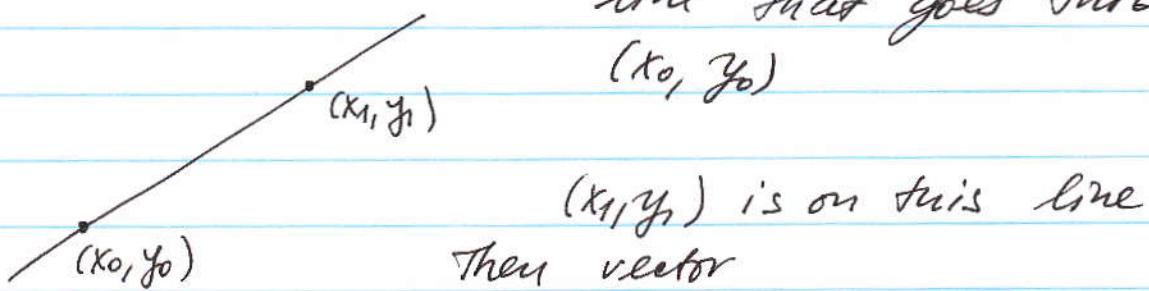
$$\text{at } (x_0, y_0) \text{ we have } z_x(x_0, y_0) = \frac{\partial z}{\partial x}(x_0, y_0)$$

$$z_y(x_0, y_0) = \frac{\partial z}{\partial y}(x_0, y_0)$$

Let m be a slope of the tangent line
 to level curve at (x_0, y_0) . Then

$$m = -\frac{z_x(x_0, y_0)}{z_y(x_0, y_0)}$$

$y - y_0 = m(x - x_0)$: equation of the tangent line that goes through (x_0, y_0)



(x_1, y_1) is on this line
Then vector

$\vec{v} = (x_1 - x_0, y_1 - y_0)$ is on the line, it is parallel to the tangent line

since (x_1, y_1) is on the line \Rightarrow

$y_1 - y_0 = m(x_1 - x_0)$ is satisfied
then

$$\vec{v} = (x_1 - x_0, y_1 - y_0) = (x_1 - x_0, m(x_1 - x_0))$$

Finally, we want to show that $\nabla z(x_0) \perp \vec{v}$

$$\nabla z(x_0) = (z_x(x_0, y_0), z_y(x_0, y_0))$$

then

$$\begin{aligned} \nabla z(x_0) \cdot \vec{v} &= (z_x(x_0, y_0), z_y(x_0, y_0)) \cdot (x_1 - x_0, m(x_1 - x_0)) \\ &= z_x(x_0, y_0) \cdot (x_1 - x_0) + z_y(x_0, y_0) \cdot m(x_1 - x_0) \quad \blacksquare \end{aligned}$$

$$\text{but } m = -\frac{z_x(x_0, y_0)}{z_y(x_0, y_0)}$$

$$\blacksquare z_x(x_0, y_0)(x_1 - x_0) + z_y(x_0, y_0) \cdot \left(-\frac{z_x(x_0, y_0)}{z_y(x_0, y_0)} \right) \cdot (x_1 - x_0) = 0$$

Differentiate implicitly to find $\frac{dy}{dx}$

Ex $z(x, y) = x^4 + y^2 + 2xy$ at (x_0, y_0)

level curve: $x^4 + y^2 + 2xy = k \quad | \quad \frac{d}{dx}$

use
product
rule
and chain
rule

$$4x^3 + 2y \cdot \frac{dy}{dx} + 2y + 2x \frac{dy}{dx} = 0 \quad \frac{dx}{dx} = 1$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}$$

$$(2y + 2x) \frac{dy}{dx} = -4x^3 - 2y$$

$$\frac{dy}{dx} = -\frac{(4x^3 + 2y)}{2y + 2x}$$

$$m = \frac{dy}{dx}(x_0, y_0) = -\frac{4x_0^3 + 2y_0}{2y_0 + 2x_0}$$

slope
of tangent
line

$$\vec{v} = (x_1 - x_0, y_1 - y_0) = (x_1 - x_0, m(x_1 - x_0)) \text{ as before}$$

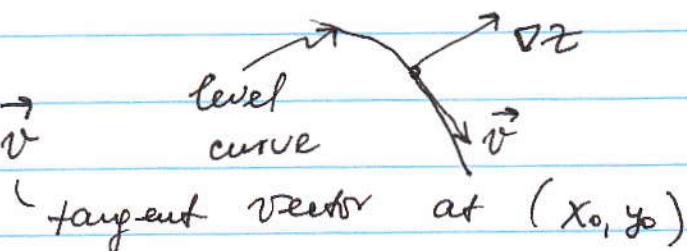
We computed $\nabla z(x_0, y_0) = (4x_0^3 + 2y_0, 2y_0 + 2x_0)$

$$\nabla z(x_0, y_0) \cdot \vec{v} = (4x_0^3 + 2y_0, 2y_0 + 2x_0) \cdot (x_1 - x_0, m(x_1 - x_0))$$

$$= (4x_0^3 + 2y_0)(x_1 - x_0) + (2y_0 + 2x_0) \left(-\frac{4x_0^3 + 2y_0}{2y_0 + 2x_0} (x_1 - x_0) \right) = 0$$

$$= 0$$

$$\Rightarrow \nabla z(x_0, y_0) \perp \vec{v}$$



1/16/2013

Gradients, constraints and Optimization

Since we study optimization problems (i.e. minimizing or maximizing some functions) gradients are very important since they give direction of the steepest ascent of a function, that can be used in maximization problem. Similarly, negation of gradient can be used to minimize the function.

Def let $g(\vec{x}) \leq b$ be a constraint in an optimization problem. If at point $\vec{x}_0 \in \mathbb{R}^n$, $g(\vec{x}_0) = b$, then the constraint is called binding.

Clearly, a constraint $h(\vec{x}) = r$ is automatically binding.

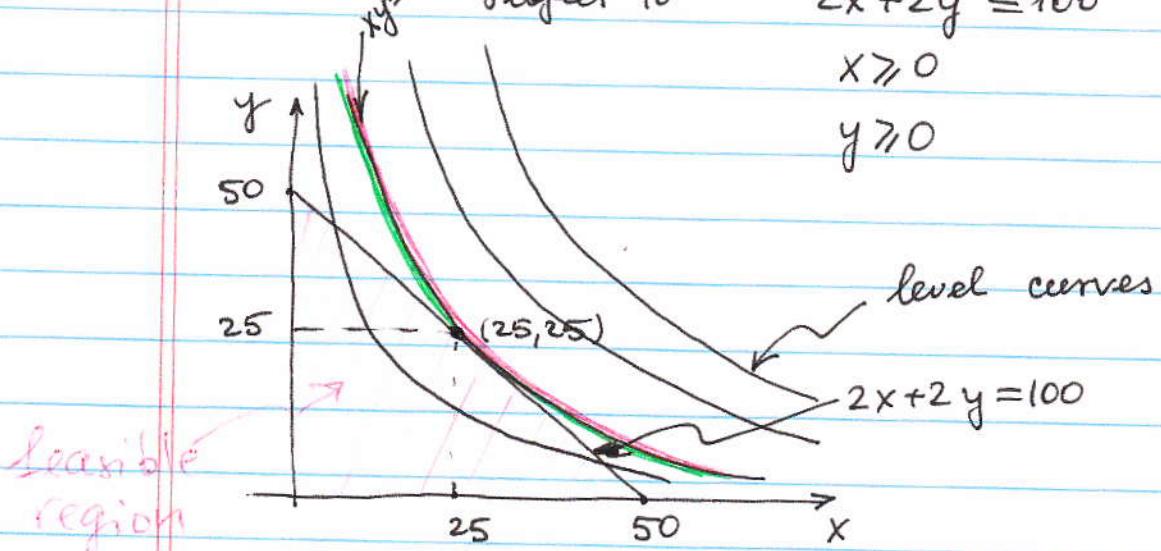
Example (Continuation of the Goat Pen ex)
Let's look at the level curves of the objective function and their relationship to constraints at the point of optimality (25, 25).

$$A(x, y) = xy \quad A = A_{\max} \text{ when } x=y=25 \\ \Rightarrow \text{point of optimality is } (25, 25).$$

The level curves of the objective function are

$$xy = k \Rightarrow y = \frac{k}{x} : \text{hyperbolas}$$

Recall : max $A(x,y) = xy$
 subject to $2x+2y \leq 100$
 $x \geq 0$
 $y \geq 0$



Note: at the point of optimality, the level curve $xy = 625$ is tangent to the equation (constraint) $2x+2y=100$

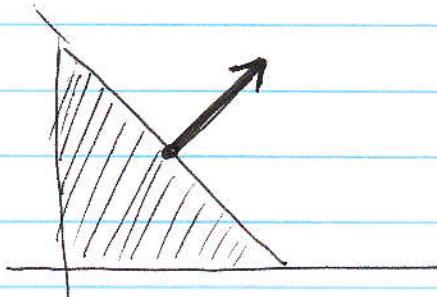
The level curve of the objective function is tangent to the binding constraint.

Now, let's look at the gradient of objective function $A(x,y) = xy$

$$\nabla A = (y, x) \quad \nabla A \Big|_{(25,25)} = (25, 25)$$

Now let's look at the gradient of the binding constraint: $2x+2y=100$. Its gradient is $(2, 2)$.

We can see that gradient of the binding constraint is a scaled version of the gradient of the objective function



Gradients of the binding constraints and objective function are scaled versions of each other at the point of optimality.

Note: $\nabla A = (25, 25)$: points in the direction of increasing area A

Simple linear Programming Problems (S1.1)

When both objective and all constraints in optimization problem (*) (see lecture 1/9/2013) are linear functions, then the optimization problem is called a linear programming problem. This has the general form:

$$\max z(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$$

$$\text{subject to } a_{11} x_1 + \dots + a_{1n} x_n \leq b_1$$

$$\dots \dots \dots \dots \dots \dots \dots$$

$$a_{m1} x_1 + \dots + a_{mn} x_n \leq b_m$$

$$\dots \dots \dots \dots \dots \dots \dots$$

$$b_{m1} x_1 + \dots + b_{mn} x_n = r_m$$

$$\dots \dots \dots \dots \dots \dots \dots$$

$$b_{r1} x_1 + \dots + b_{rn} x_n = r_r$$

Def A function $z \in \mathbb{R}^n \rightarrow \mathbb{R}$ is linear if there are constants $c_1, c_2, \dots, c_n \in \mathbb{R}$ such that

$$z(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$$

A linear function $z \in \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies the following (linearity) conditions:

$$(1) \quad z(\vec{x}_1 + \vec{x}_2) = z(\vec{x}_1) + z(\vec{x}_2)$$

$$(2) \quad z(\alpha \vec{x}) = \alpha z(\vec{x}), \quad \alpha \in \mathbb{R}$$

For now, we will concentrate on linear programming problems for functions of two variables. For this case, we will develop a graphical method for identifying optimal solutions, which we will generalize for higher dimensions, i.e. for functions of more than two variables.

Ex Consider the problem of a toy company that produces toy planes and toy boats. The toy company can sell its planes for \$10 and its boats for \$8. It costs \$3 in raw materials to make a plane and \$2 to make a boat. A plane requires 3 hours to make and 1 hour to finish while a boat requires 1 hour to make and 2 hours to finish.

The toy company knows that it can sell at most 35 planes per week. Further, given the number of workers, the company cannot spend anymore than 160 hours per week finishing toys and 120 hours per week making toys. The company wishes to maximize the profit by choosing how many of each toy to produce.

Let x_1 be the number of planes the company will produce and let x_2 be the # of boats.

The profit for each plane is $\$10 - \$3 = \$7$, profit for each boat is $\$8 - \$2 = \$6$.
 \Rightarrow total profit is

$$z(x_1, x_2) = 7x_1 + 6x_2$$

Company can spend no more than 120 hour per week making toys, and since a plane takes 3 h to make and a boat takes 1 h to make, we have:

$$3 \cdot x_1 + 1 \cdot x_2 \leq 120$$

Similarly, company can spend at most 160 hours finishing toys, and since it takes

1h to finish a plane and 2 hours to finish a boat, we have

$$1 \cdot x_1 + 2 \cdot x_2 \leq 160$$

We also know $x_1 \leq 35$ and $x_1 \geq 0, x_2 \geq 0$

→ Complete linear programming problem is

$$\text{max } z(x_1, x_2) = 7x_1 + 6x_2$$

$$\text{subject to } 3x_1 + x_2 \leq 120$$

$$x_1 + 2x_2 \leq 160$$

$$x_1 \leq 35$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

1/18/2013

Modeling Assumptions in linear Programming

A problem can be phrased as a linear programming problem if the following assumptions are satisfied:

1. Proportionality assumption: the contribution to the objective function and left-hand-sides of each constraint by each decision variable (x_1, \dots, x_n) is proportional to the value of the decision variable.
2. Additivity assumption: the contribution to the objective function and left-hand-side of each constraint by any decision variable x_i ($i=1, \dots, n$) is completely independent of any other decision variable.
3. Divisibility assumption: the quantities that are represented by decision variables are infinitely divisible (i.e. fractional answer makes sense).
4. Certainty Assumption: the coefficients in the objective function and constraints are known with certainty.

Assumptions 1 and 2 are consequences of linearity of the objective function and constraints.

Divisibility assumption says that answers can be fractions. If we insist on having integer answers, then the problem is called an integer programming problem.

Especially in large problems, when, say, the answer is 1067.3 planes to make, the answer is rounded to the nearest integer: 1067 to make sense.

If certainty assumption is relaxed, the problem may be considered as a stochastic programming problem.

If the objective function is quadratic function but constraints are still linear, then we have a quadratic programming problem.

In general, we have a nonlinear programming problem.

Graphically Solving linear Program

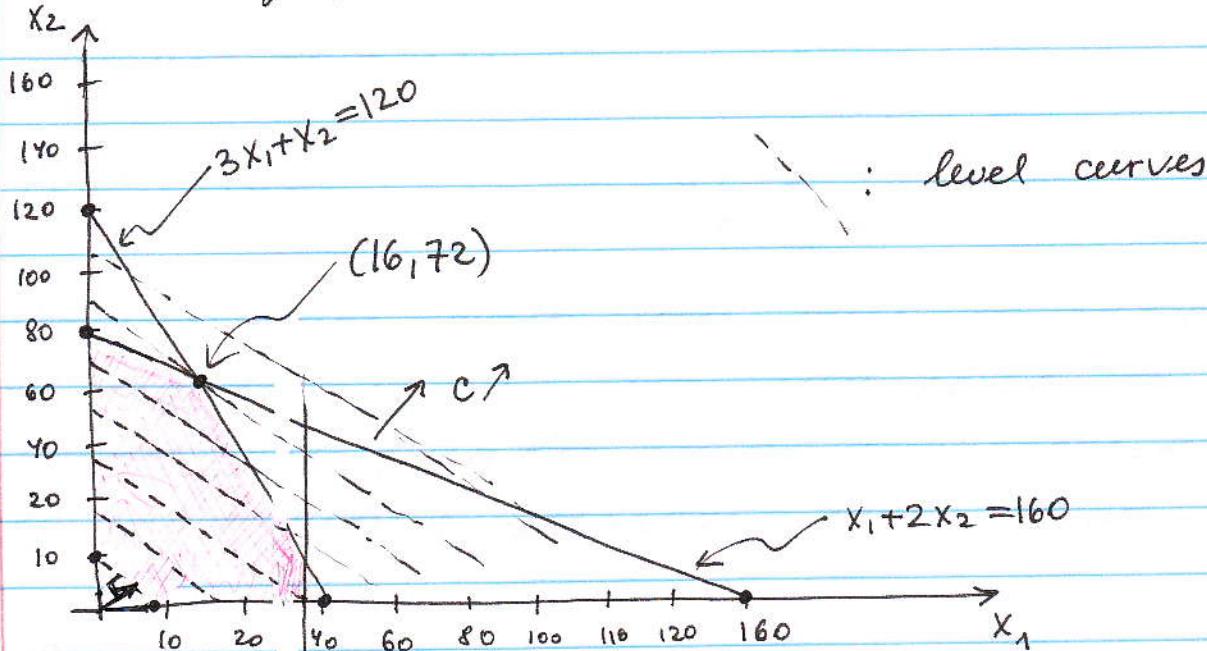
Problems with Two Variables (Bounded Case)

Linear programming problem in this case can be solved by plotting feasible region with the level sets of the objective function.

Ex Toy Maker problem (Cont'd)

$$\begin{aligned} \text{max } z(x_1, x_2) &= 7x_1 + 6x_2 \\ \text{subject to } &3x_1 + x_2 \leq 120 \\ &x_1 + 2x_2 \leq 160 \\ &x_1 \leq 35 \\ &x_1 \geq 0 \\ &x_2 \geq 0 \end{aligned}$$

First we graph the feasible region.



This point maximizes $z(x_1, x_2) = 7x_1 + 6x_2$

Point $(16, 72)$ is also a point of intersection of two constraints:

$$3x_1 + x_2 \leq 120$$

and

$$x_1 + 2x_2 \leq 160$$

At $(16, 72)$, the equalities are satisfied, i.e. $3x_1 + x_2 = 120$, $x_1 + 2x_2 = 160$, hence these constraints are binding. Other constraints are not binding.

Solution of the optimisation problem, as we will also see later, is in the corner formed by binding constraints.