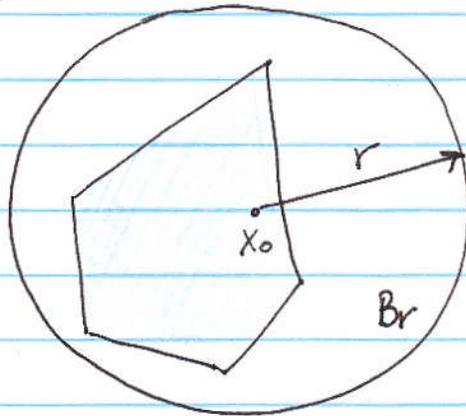


1/23/2013

## Formalizing the Graphical Method

The example about the toy factory was an example of a linear programming problem which had a unique solution and this problem also had a bounded feasible region



Region is bounded if it can be placed inside a disk (ball)  $B_r$  with center at some pt  $x_0$  and finite radius  $r$

In general, the following outcomes in solving a linear programming problem are possible:

1. The linear programming problem has a unique solution (we have seen this).
2. There are infinitely many alternative optimal solutions.
3. There is no solution and the problem's objective function  $\rightarrow +\infty$  when a maximization problem is solved (or  $\rightarrow -\infty$  for minimization problem)
4. There is no solution at all.

## Algorithm for solving a linear programming problem graphically (bounded domain, unique solution).

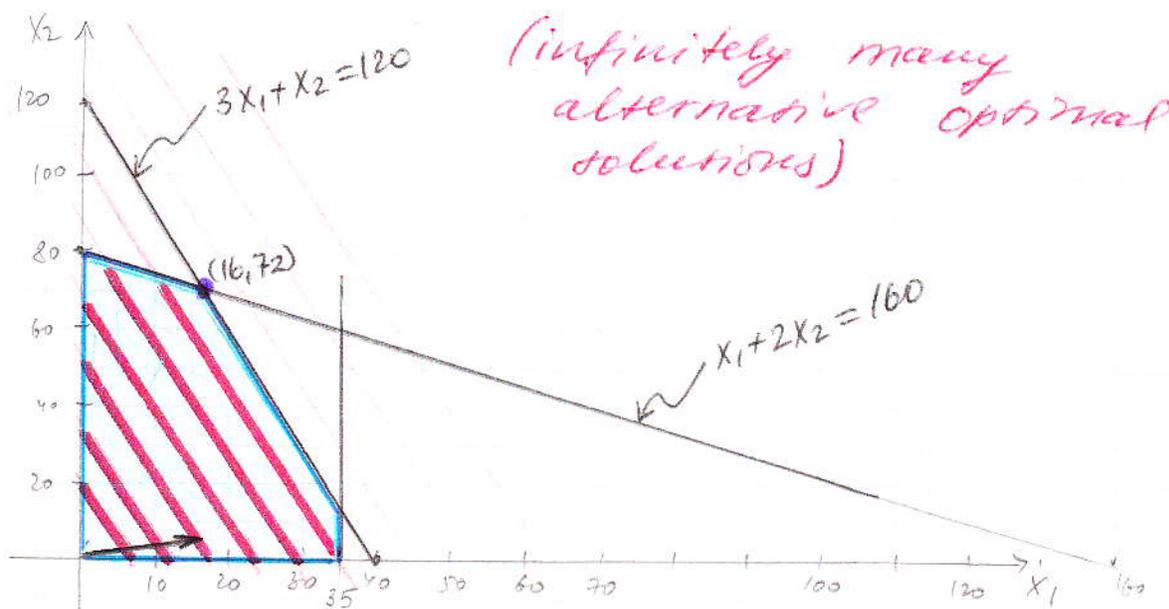
1. Plot the feasible region defined by constraints.
2. Plot level curves (level sets) of the objective function
3. Find the gradient of the objective function.
4. For a maximization problem, identify the level set corresponding to the greatest value of the objective function that intersects the feasible region (follow in the direction of gradient). For min. problem, follow the direction of  $-\nabla z$ . The point is at the corner of the feasible domain.
5. This point at the corner is the solution of the optimization problem.

### Alternative Optimal Solutions

**Ex 1** Example: modify the toy company problem.  
Let's change profit of selling a plane from \$7 each to \$18 each.

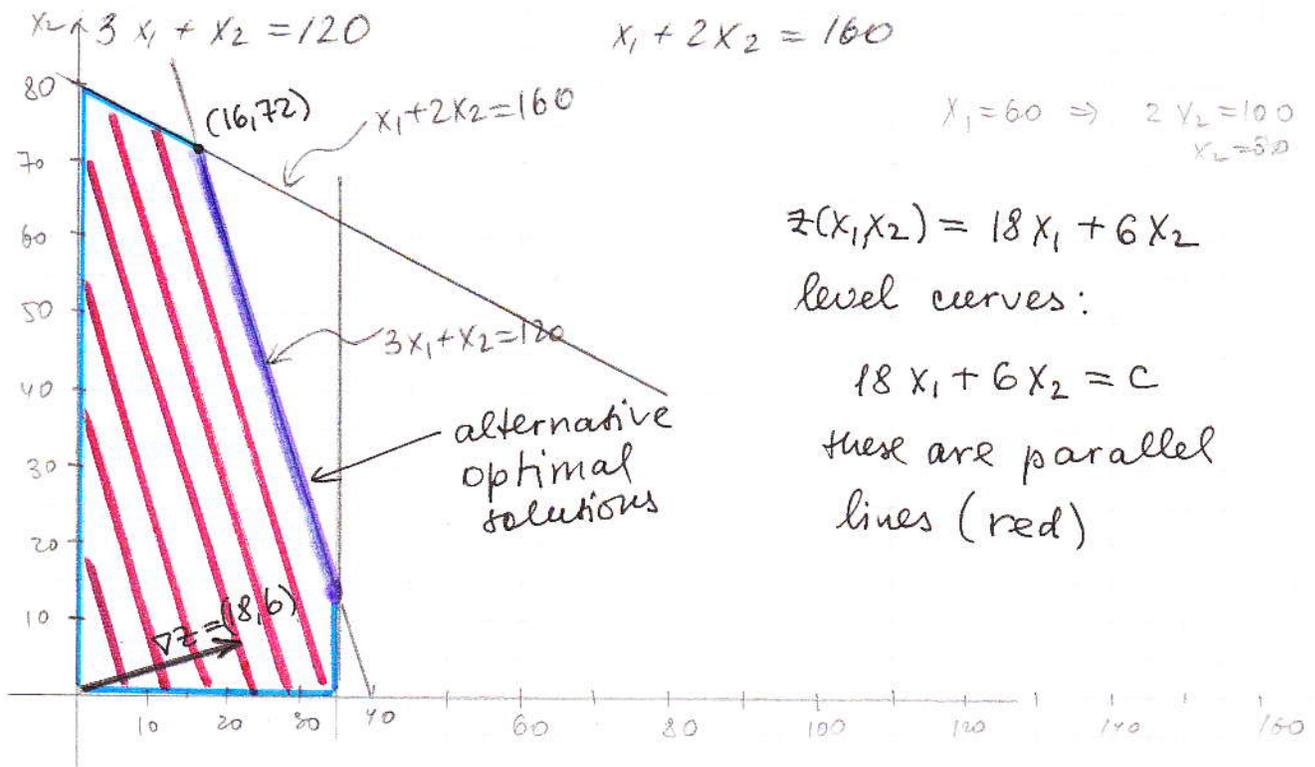
$$\begin{aligned} \max z(x_1, x_2) &= 18x_1 + 6x_2 \\ \text{subject to} \quad & 3x_1 + x_2 \leq 120 \\ & x_1 + 2x_2 \leq 160 \\ & x_1 \leq 35, \quad x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

Ex 1



(infinitely many alternative optimal solutions)

or (just a square scale)



$18x_1 + 6x_2 = C$

$6(3x_1 + x_2) = C$  || to boundary side  $3x_1 + x_2 = 120$   
 $\Rightarrow$  all level curves are || to the side of feasible domain  $3x_1 + x_2 = 120$

$\nabla z = (18, 6) = 6(3, 1)$

In this problem instead of one point, we get infinitely many points located on one side of feasible domain defined by

$$(x_1, x_2) \quad 16 \leq x_1 \leq 35 \quad (1)$$

$$3x_1 + x_2 = 120$$

Any point  $(x_1^*, x_2^*)$  from (1) will be an optimal solution, i.e.

$$z(x_1^*, x_2^*) \geq z(x_1, x_2), \quad (x_1, x_2) \text{ is inside feasible domain}$$

### Problems with no solutions

It may be possible that your feasible domain is empty. In this case, the problem is over constrained.

Ex 2 Ex

$$\max z(x_1, x_2) = 3x_1 + 2x_2$$

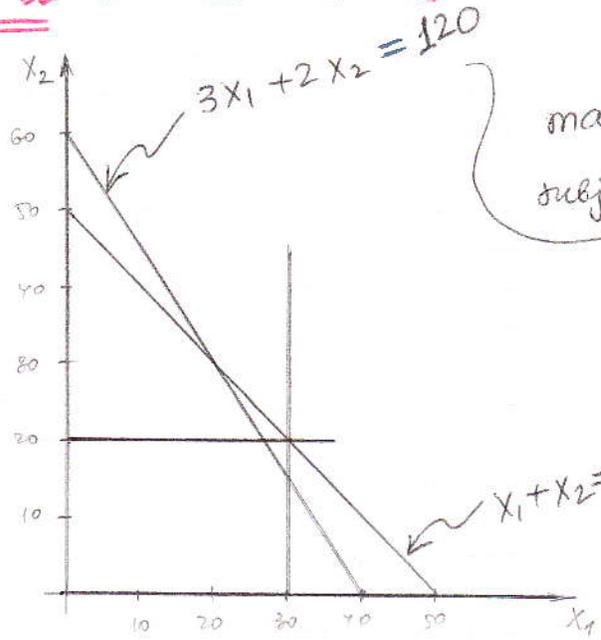
subject to  $\frac{1}{70}x_1 + \frac{1}{60}x_2 \leq 1 \quad (\Leftrightarrow) \quad 3x_1 + 2x_2 \leq 120$

$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \leq 1 \quad (\Leftrightarrow) \quad x_1 + x_2 \leq 50$$

$$x_1 \geq 30$$

$$x_2 \geq 20$$

EX 2 (no solutions bounded domain)



$$\max z(x_1, x_2) = 3x_1 + 2x_2$$

$$\text{subject to } \frac{1}{40}x_1 + \frac{1}{60}x_2 \leq 1$$

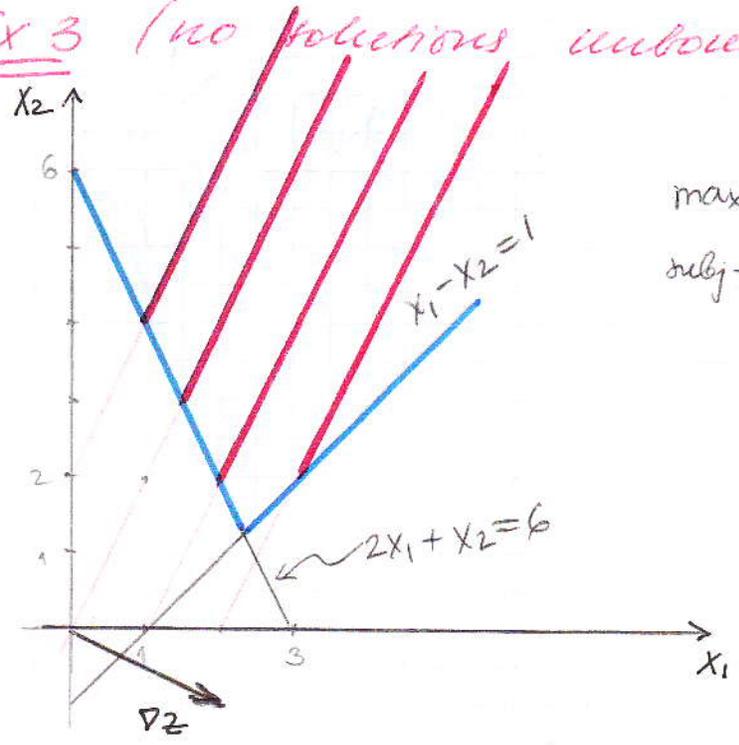
$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \leq 1$$

$$x_1 \geq 30$$

$$x_2 \geq 20$$

Feasible region is empty.

EX 3 (no solutions unbounded domain)



$$\max z(x_1, x_2) = 2x_1 - x_2$$

$$\text{subject to } x_1 - x_2 \leq 1$$

$$2x_1 + x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

level curves:  $2x_1 - x_2 = c \Rightarrow x_2 = 2x_1 - c$

$\nabla z(x_1, x_2) = (2, -1)$

In this case, there <sup>will</sup> always be an intersection point of a level curve and feasible domain for any value of  $c$  we choose. Hence, values of  $z(x_1, x_2) \rightarrow +\infty$ . No solution.

## Problems with unbounded feasible regions

Ex 3

Ex Consider the linear programming problem

$$\max z(x_1, x_2) = 2x_1 - x_2$$

$$\text{subject to } x_1 - x_2 \leq 1$$

$$2x_1 + x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

$z(x_1, x_2) \rightarrow +\infty$ . No optimal solution

Ex 4

Ex Modify objective function in the previous case.

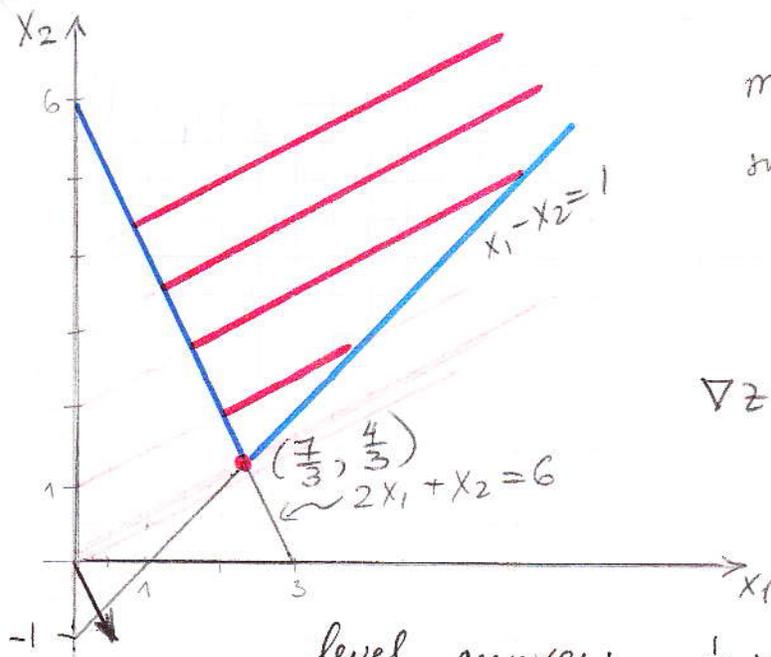
$$\max z(x_1, x_2) = \frac{1}{2} x_1 - x_2$$

$$\text{subject to } x_1 - x_2 \leq 1$$

$$2x_1 + x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

### Ex 4



$$\max z(x_1, x_2) = \frac{1}{2} x_1 - x_2$$

$$\text{subject to } x_1 - x_2 \leq 1$$

$$2x_1 + x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

$$\nabla z(x_1, x_2) = \left(\frac{1}{2}, -1\right)$$

level curves:  $\frac{1}{2} x_1 - x_2 = c$

$$x_2 = \frac{1}{2} x_1 - c$$

Unique solution  $\left(\frac{7}{3}, \frac{4}{3}\right)$ .

Note: in this example,  $\nabla z$  points in the direction opposite to direction in which feasible domain is unbounded.

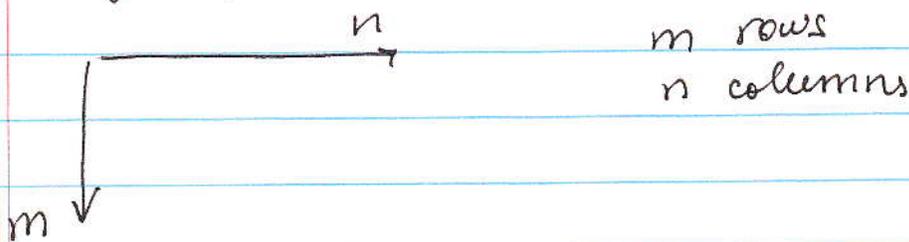
1/25/2013

## Matrices, Linear Algebra and Linear Programming

Let  $\vec{x}, \vec{y}$  be vectors from  $\mathbb{R}^n$ . We will denote by  $\vec{x} \cdot \vec{y}$  their dot product.

### Matrices

Recall that  $m \times n$  matrix is a rectangular array of numbers from  $\mathbb{R}$



$A$ : matrix of size  $m \times n$

$A_{ij}$ : element of  $A$  in  $i^{\text{th}}$  row and  $j^{\text{th}}$  column

If  $m=n$ , matrix  $A$  is square.

If  $A, B$  are two  $m \times n$  matrices, then their sum  $C = A + B$  is an  $m \times n$  matrix such that

$$C_{ij} = A_{ij} + B_{ij}$$

$$\underline{\underline{\text{Ex}}} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Def Matrix of size  $m \times 1$  is a column vector.  
 Matrix of size  $1 \times n$  is a row vector.

Usually  $\vec{x}$  will stand for a column vector unless it states otherwise.

$A(i,:)$   $A_{i,:}$  : notation for  $i^{\text{th}}$  row of matrix  $A$   
 $A(:,j)$  :  $j^{\text{th}}$  column of matrix  $A$

Def Let  $A \in \mathbb{R}^{m \times n}$ ,  $B \in \mathbb{R}^{n \times p}$ . Then the product of  $A$  and  $B$  is matrix  $C \in \mathbb{R}^{m \times p}$  such that

$$C_{ij} = A_{i,:} \cdot B_{:,j}$$

$A_{i,:}$  is  $1 \times n$  <sup>row</sup> vector

$B_{:,j}$  is  $n \times 1$  column vector

$$\underline{\underline{\text{Ex}}} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Multiplication of matrices is not commutative in general, i.e.  $AB \neq BA$

Def If  $A \in \mathbb{R}^{m \times n}$  is an  $m \times n$  matrix, then its transpose, denoted by  $A^T$ , is a matrix of size  $n \times m$  such that

$$A_{ij}^T = A_{ji}$$

Ex

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Transpose matrices are used to write dot product as multiplication of vectors.

For example, if  $\vec{x}$  is  $n \times 1$  column vector, then  $\vec{x}^T$  is  $1 \times n$  row vector.

Let  $\vec{y}$  be  $n \times 1$  column vector.

Then

$$\vec{x} \cdot \vec{y} = \vec{x}^T \cdot \vec{y}$$

Other properties of transpose matrices:

$$1. (A+B)^T = A^T + B^T$$

$$2. (AB)^T = B^T \cdot A^T$$

Let  $A$  and  $B$  be two matrices with the same # of rows, e.g.  $A \in \mathbb{R}^{m \times n}$ ,  $B \in \mathbb{R}^{m \times p}$ .

Then the augmented matrix  $[A|B]$  is

$$\left( \begin{array}{cccc|cccc} a_{11} & a_{12} & \dots & a_{1n} & b_{11} & b_{12} & \dots & b_{1p} \\ a_{21} & a_{22} & \dots & a_{2n} & b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_{m1} & b_{m2} & \dots & b_{mp} \end{array} \right) : \begin{array}{l} m \times (n+p) \\ \text{matrix} \end{array}$$

Ex

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$$[A | \vec{b}] = \left[ \begin{array}{cc|c} 1 & 2 & 7 \\ 3 & 4 & 8 \end{array} \right]$$

Note By analogy, we can define augmented matrix  $\left[ \begin{array}{c} A \\ B \end{array} \right]$ . This is NOT division.

Matrices A, B have the same # of columns.

Special matrices and vectors

Identity matrix of  $n \times n$  size

$$I_n = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \dots & \\ 0 & & & 1 \end{pmatrix} \equiv I$$

$$AI = IA = A$$

5

Def The standard vector basis in  $\mathbb{R}^n$  is formed by vectors  $\vec{e}_i \in \mathbb{R}^n$  such that

$$\vec{e}_i = (0, \dots, \underbrace{0}_{i-1}, \underbrace{1}_i, 0, \dots, 0) = I_i.$$

The vector  $\vec{e} \in \mathbb{R}^n$  is the one vector:  $\vec{e}^T = (1, 1, \dots, 1)$   
The zero vector  $\vec{0} = (0, 0, \dots, 0)$ .

Exercise Let  $\vec{x} \in \mathbb{R}^n$  (column vector)  
One can show that for vector

$$\vec{y} = \frac{\vec{x}}{\vec{e}^T \vec{x}}$$

the following  $\vec{e}^T \vec{y} = \vec{y}^T \vec{e} = 1$

Indeed,

$$\vec{e}^T \vec{y} = \frac{\vec{e}^T \vec{x}}{\vec{e}^T \vec{x}} = 1 \quad \checkmark$$

$\vec{e}^T \cdot \vec{x}$  is dot product  $\vec{e} \cdot \vec{x}$ , is the number

### Matrices and Linear Programming Expression

Consider a system of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

It can be written in a compact matrix form:

$$A\vec{x} = \vec{b}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A system can also be given with inequalities. Then we could have

$$A\vec{x} \leq \vec{b}$$

Using this representation, a general linear programming problem can be written as

$$\begin{aligned} \max z(\vec{x}) &= \vec{c}^T \vec{x} \\ \text{subject to} \quad & A\vec{x} \leq \vec{b} \\ & H\vec{x} = \vec{r} \end{aligned}$$

Def a maximization linear programming problem in the canonical form is

$$\begin{aligned} \max z(\vec{x}) &= \vec{c}^T \vec{x} \\ \text{subject to} \quad & A\vec{x} \leq \vec{b} \\ & \vec{x} \geq 0 \end{aligned}$$

Def A minimization linear programming problem in the canonical form is

$$\begin{aligned} \min z(\vec{x}) &= \vec{c}^T \vec{x} \\ \text{subject to } A\vec{x} &\geq \vec{b} \\ \vec{x} &\geq 0 \end{aligned}$$