

2/22/2013

Caratheodory Characterization

Theorem

Lemma 1 The polyhedral set defined by

$$P = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$$

has a finite, non-zero number of extreme points (assuming that A is a non-empty matrix).

Lemma 2 Let P be a non-empty polyhedral set. Then the set of directions of P is empty if and only if P is bounded.

Lemma 3 Let P be a non-empty unbounded set. Then the number of extreme directions of P is finite and non-zero.

Theorem (Caratheodory Characterization Thm)

Let P be a non-empty unbounded polyhedral set defined by

$$P = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$$

Suppose that P has extreme points x_1, x_2, \dots, x_k and extreme directions d_1, d_2, \dots, d_ℓ . If $x \in P$, then there exist constants $\lambda_1, \lambda_2, \dots, \lambda_k$ and $\mu_1, \mu_2, \dots, \mu_\ell$ such that

$$x = \sum_{i=1}^k \lambda_i x_i + \sum_{j=1}^\ell \mu_j d_j$$

$$\sum_{i=1}^k \lambda_i = 1$$

$$\lambda_i \geq 0, i=1, \dots, k$$

$$\mu_j \geq 0, j=1, \dots, \ell$$

This means that any $x \in P$ can be written as a convex combination of extreme points plus positive combination of extreme directions.

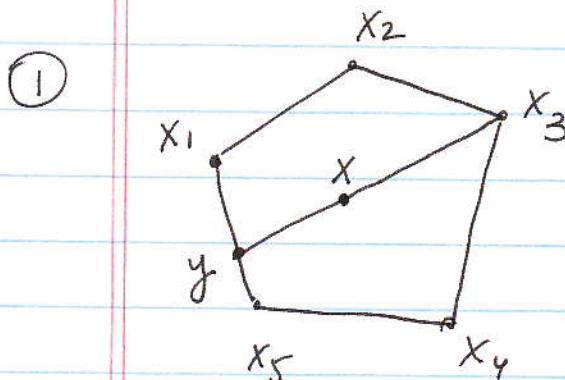
Note

If P is a bounded set, then it has no directions and the rest of the theorem still holds, i.e. for any $x \in P$, there exist constants $\lambda_1, \lambda_2, \dots, \lambda_k$ such that

$$x = \sum_{i=1}^k \lambda_i x_i,$$

$$\sum_{i=1}^k \lambda_i = 1, \quad \lambda_i \geq 0, i=1, \dots, k$$

Ex These examples illustrate how one could express any point from inside P as a convex combination of extreme pts and positive combination (if any) of extreme directions



P is a bounded polyhedral set

$$x = 2y + (1-2)x_3 : \text{convex comb.}$$

$$y = \mu x_1 + (1-\mu)x_5 : \text{convex comb.}$$

Then

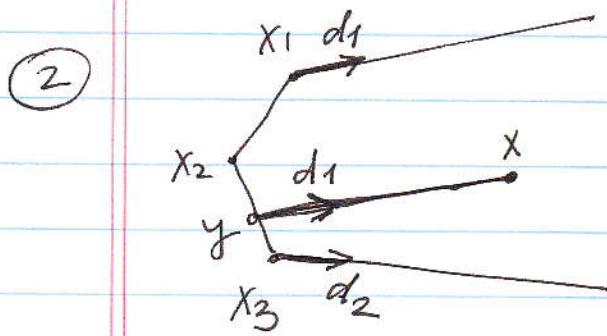
$$x = 2[\mu x_1 + (1-\mu)x_5] + (1-2)x_3$$

$$= 2\mu x_1 + 2(1-\mu)x_5 + (1-2)x_3$$

This is a convex combination of x_1, \dots, x_5 (coefficient of x_2 is 0, coef. of x_4 is also 0)

$$2\mu + 2(1-\mu) + (1-2) = 2[\mu + (1-\mu)] + 1-2 =$$

$$= 2 + 1 - 2 = 1$$



P is unbounded set

P has extreme points
 x_1, x_2, x_3 and extreme

directions d_1, d_2

segment connecting pts x and y is in
 d_1 direction \Rightarrow

$$x = y + \lambda d_1, \quad \lambda \geq 0$$

$y = \mu x_2 + (1-\mu)x_3$: convex combination of
 x_2, x_3

$$\therefore x = \underbrace{\mu x_2 + (1-\mu)x_3}_{\text{convex comb. with coefficient of } x_1 \text{ being 0}} + \underbrace{\lambda d_1}_{\text{positive combination}}$$

The Simplex Method (Chapter 3)

We know that if there is an optimal solution, then it occurs at an extreme point.

We will assume that matrix $A \in \mathbb{R}^{m \times n}$ has full rank and $b \in \mathbb{R}^m$ and let

$$X = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\} : \text{polyhedral set}$$

We would like to maximize (or minimize) - later an objective function

$\tilde{z}(x_1, \dots, x_n) = c^T x$, where $c, x \in \mathbb{R}^n$.

We consider the following LP problem:

$$\text{Problem P} \quad \left\{ \begin{array}{l} \max c^T x \\ \text{s.t. } Ax \leq b \\ x \geq 0 \end{array} \right.$$

Think about
an optimal
solution

Thm If Problem P has an optimal solution, then Problem has an optimal extreme point solution.

Pf We will use Caratheodory Thm that says that any point of X can be written as

$$x = \sum_{i=1}^k z_i x_i + \sum_{j=1}^k \mu_j d_j$$

where x_1, \dots, x_n are extreme pts of X and d_1, d_2, \dots, d_l are extreme directions of X

$$\sum_{i=1}^k \lambda_i = 1, \quad \lambda_i \geq 0, \quad i=1, \dots, k, \quad \mu_j \geq 0, \quad j=1, \dots, l.$$

Now we can rewrite problem P as

$$\max \underbrace{\sum_{i=1}^k z_i c^T x_i}_{\text{L}} + \underbrace{\sum_{j=1}^l \mu_j c^T d_j}_{\text{R}} \quad z = c^T x$$

$$\text{s.t. } \sum_{i=1}^k \gamma_i = 1, \quad \gamma_i \geq 0, \quad \mu_j \geq 0, \quad i=1, \dots, k \\ \quad \quad \quad j=1, \dots, l$$

If for some j , $C^T d_j > 0$, then we can choose μ_j as large as we like and make objective function larger and larger. Then in this case we will have no finite solution.

Hence, we will assume that all $C^T d_j \leq 0$. Then to maximize objective function we can choose all $\mu_j = 0$, $j=1, \dots, l$.

2/25/2013

Thm about an optimal solution (Cont'd)

$$\max z = \sum_{i=1}^k z_i c^T x_i + \sum_{j=1}^l y_j \underbrace{c^T d_j}_{\leq 0} = \sum_{i=1}^k z_i c^T x_i$$

convex
comb.

We know that the set of extreme points x_1, x_2, \dots, x_k is finite, $z_i \geq 0$, $\sum_{i=1}^k z_i = 1$.

To maximize the objective function z we can set $z_p = 1$ if $c^T x_p$ has the largest value among all possible values $c^T x_i$, $i=1, \dots, k$. This is clearly solution of our linear programming problem and we can see that this solution is achieved at an extreme pt. Conclusion, if problem P has an optimal, it must be an extreme point optimal solution.

Corollary 1. Problem P has a finite solution if and only if $c^T d_j \leq 0$ for all $j=1, \dots, l$ where d_1, d_2, \dots, d_l are extreme directions of X .

Pf This is implicit in the proof of Theorem.

Note In HW #6, modify the proof of Thm from $\max z$ to $\min z$.

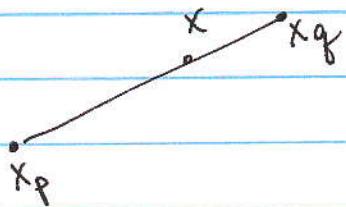
Corollary 2 Problem P has alternative optimal solutions if there are at least two extreme points x_p and x_g so that $c^T x_p = c^T x_g$ and so that x_p is the extreme point solution to the LP problem (so as x_g).

Proof suppose that x_p is the optimal point solution of problem P identified in the proof of the previous Thm. Suppose that x_g is another optimal solution with ^{extreme}

$c^T x_p = c^T x_g$. Both x_p and x_g are extreme pts of X , X is a convex set. Hence every point

$$x = \begin{matrix} z \\ 1 \end{matrix} x_p + \begin{matrix} 1-z \\ 1 \end{matrix} x_g \in X$$

convex combination
of x_p and x_g



Then the value of the objective function at such x is

$$z = c^T x = c^T \left(\begin{matrix} z \\ 1 \end{matrix} x_p + \begin{matrix} 1-z \\ 1 \end{matrix} x_g \right) =$$

$$= \underbrace{z c^T x_p}_{c^T x_p} + \underbrace{(1-z)c^T x_g}_{\parallel} = z c^T x_p + (1-z)0^T x_p =$$

$$= c^T x_p$$

But x_p is an extreme pt optimal solution \Rightarrow any pt connecting x_p and x_q gives the same value of z . Hence, pts on the segment connecting x_p and x_q give alternative optimal solutions.

Exercise (HW #6)

Let $X = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$ and suppose d_1, d_2, \dots, d_ℓ are the extreme directions of X (assuming it has any). Show that the problem

$$\begin{aligned} & \min c^\top x \\ \text{s.t. } & Ax \leq b \\ & x \geq 0 \end{aligned}$$

has a finite optimal solution iff $c^\top d_j \geq 0$, $j=1, \dots, \ell$. [Hint: modify the proof of Corollary 1. [Characterization of the proof of Corollary 1].]

Algorithmic Characterization of Extreme Points

We showed that if an optimal solution exists, it has to occur at an extreme point. We need to find a way how to find extreme points.

let

$$X = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\} : \text{polyhedral set}$$

$m \leq n$ $A \in \mathbb{R}^{m \times n}$, we assume $\text{rank}(A) = m$, i.e.
A has full rank.

We can split matrix $A = [B \mid N]$

basic matrix contains m lin. indep. columns of A
non-basic matrix

Matrix B is also called basis. $B \in \mathbb{R}^{m \times m}$

$$N \in \mathbb{R}^{m \times (n-m)}$$

We wrote solution $x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$

x_B : basic variables or dependent variables

x_N : non-basic variables or independent variables

$$x_B = B^{-1}b - B^{-1}N x_N$$

We know that B is invertible since A has full rank m.

Setting $x_N = 0$, we get

$$x_B = B^{-1}b: \underline{\text{basic solution}}$$

Any basic solution satisfies $AX=b$ but it may not satisfy the constraints $x \geq 0$.

Def If $x_B = B^{-1}b$ and $x_N = 0$ is a basic solution to $AX=b$ and $x_B \geq 0$, then (x_B, x_N) is called basic feasible solution.

Thm 2 Every basic feasible solution is an extreme pt of X . Likewise, every extreme point is characterized by a basic feasible solution of $AX=b$, $x \geq 0$.

Recall result:

Thm 3

Thm For a polyhedral set

$$X = \{x \in \mathbb{R}^n : Ax \leq b\}, \quad A \in \mathbb{R}^{m \times n}$$

Point $x_0 \in X$ is an extreme pt of X iff x_0 is an intersection of n linearly independent hyperplanes defining X .

Proof of Thm 2

2/27/2013

see
lecture
2/25/2013

Proof of Thm 2 (Every basic feasible solution is an extreme pt of X and vice versa, every extreme pt can be identified with some basic feasible solution)

$$A = [B \mid N]$$

B has m lin. indep. columns
of A

$$A \in \mathbb{R}^{m \times n}, \quad x = [x_B, x_N]^T$$

Since $Ax = Bx_B + Nx_N = b$, this represents intersection of m linearly independent hyperplanes. For a basic solution, we set $x_N = 0$. Since we had constraint $x \geq 0$, condition $x_N = 0$ implies that we have $n-m$ binding constraints corresponding to components in x_N . Thus, the point $x = (x_B, x_N)$ is intersection of $m + (n-m) = n$ lin. independent hyperplanes. By Thm 3 from last lecture (last thm), this implies that $x = (x_B, x_N)$ is an extreme pt of X .

Conversely, let x be an extreme pt of X . Then it has to be an intersection of n lin. independent hyperplanes. Since x is an extreme pt of X , it is feasible. This implies that it satisfies $Ax = b$.

This accounts for m linearly independent hyperplanes. The rest $n-m$ hyperplanes have to come from the constraint $x \geq 0$, i.e. there are $n-m$ variables that are zero (or $n-m$ constraints of $x \geq 0$ are binding: $x=0$). Call these zero variable components of x_N : $x_N = 0$. Denote the rest of variables by vector x_B . This implies $A = [B \mid N]$ and $x = [x_B, x_N]$. $\Rightarrow x_B$ is a basic feasible solution.

The Simplex Algorithm - Algebraic Form

The idea of the simplex method is the following.

1. Convert the linear problem into standard form.
2. Obtain initial basic feasible solution (if possible)
3. Determine whether the basic feasible solution is optimal. If yes, stop.
4. If the current basic feasible solution is not optimal, determine which non-basic variable (zero valued variable) should become basic (become non-zero) and which basic variable (non-zero valued variable) should become non-basic (go to zero) to make the value of the

objective function better.

5. Determine whether problem is unbounded.
If yes, stop.

6. If problem doesn't seem to be unbounded at this stage, find a new feasible solution. Go to step 3.

Suppose we have a basic feasible solution $x = (x_B, x_N)^T$. We can split the cost vector c into $c = (c_B, c_N)^T$. Then the objective function is

$$(1) \quad z = c^T x = c_B^T x_B + c_N^T x_N$$

We have expression for solution x in terms of basic B and non-basic N matrices:

$$(2) \quad x_B = B^{-1}b - B^{-1}N x_N$$

Then substitute (2) into (1):

$$\begin{aligned} z &= c_B^T (B^{-1}b - B^{-1}N x_N) + c_N^T x_N = \\ &= c_B^T B^{-1}b + (c_N^T - c_B^T B^{-1}N) x_N \end{aligned}$$

Let J be the set of indices for non-basic variables. Then

$$Z(x_1, x_2, \dots, x_n) = C_B^T B^{-1} b + \sum_{j \in J} \underbrace{(c_j - C_B^T B^{-1} A_{\cdot j})}_{\begin{array}{l} \text{jth component} \\ \text{of } c_N \end{array}} x_j$$

Consider now the fact
that $x_j = 0$, $j \in J$ (i.e. $x_N = 0$)
Further, we see that

$$\frac{\partial Z}{\partial x_j} = c_j - C_B^T B^{-1} A_{\cdot j}$$

This means that if $c_j - \overbrace{C_B^T B^{-1} A_{\cdot j}} > 0$ and
we increase x_j from zero to some new
value, the value of the objective function
will also increase. For historic reasons,
we actually consider $C_B^T B^{-1} A_{\cdot j} - c_j$,
called reduced cost and denote this as

$$-\frac{\partial Z}{\partial x_j} = c_j - z_j = c_j - C_B^T B^{-1} A_{\cdot j}$$

In a maximization problem, we choose
non-basic variable x_j with negative
reduced cost function to become basic,
since in this case $\frac{\partial Z}{\partial x_j} > 0$.

3/4/2013

Simplex Method in Algebraic Form (Cont'd)

Assume that we choose variable x_j (non-basic or zero) to become basic (non-zero). Now we have to decide which basic variable (non-zero) should become basic (zero). When we swap variables, we have to make sure that none of basic variables becomes negative.

i.e. x_j will from 0 to some positive value.

Recall

$$x_B = B^{-1}b - B^{-1}N x_N$$

If we change, say, variable x_j from 0 to some positive value, only current basic variables may be affected. Let us focus on one non-basic variable x_j being non-zero, other non-basic variables are 0. Then we can write

$$\begin{aligned} x_B &= \underbrace{B^{-1}b}_{=\bar{b}} - \underbrace{\overbrace{B^{-1}A_{\cdot j}}^{\bar{a}_j} x_j}_{\text{(column of } N \text{ that corresponds to variable } x_j)} \end{aligned}$$

Denote $\bar{b} = B^{-1}b$, $\bar{a}_j = B^{-1}A_{\cdot j}$: columns of length m

Then we can write

$$x_B = \bar{b} - \bar{a}_j \cdot x_j$$

Let $\bar{b} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_m \end{bmatrix}$ and $\bar{a}_j = \begin{bmatrix} \bar{a}_{j1} \\ \bar{a}_{j2} \\ \vdots \\ \bar{a}_{jm} \end{bmatrix}$. Then

$$x_B = \begin{bmatrix} x_{B_1} \\ x_{B_2} \\ \vdots \\ x_{B_m} \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_m \end{bmatrix} - \begin{bmatrix} \bar{a}_{j1} \\ \bar{a}_{j2} \\ \vdots \\ \bar{a}_{jm} \end{bmatrix} x_j = \begin{bmatrix} \bar{b}_1 - \bar{a}_{j1} x_j \\ \bar{b}_2 - \bar{a}_{j2} x_j \\ \vdots \\ \bar{b}_m - \bar{a}_{jm} x_j \end{bmatrix}$$

We know $\bar{b}_i \geq 0$, $i=1, \dots, m$ since $B^T b$ is basic feasible solution. If $\bar{a}_{ji} < 0$, then as we increase x_j , $\bar{b}_i - \bar{a}_{ji} x_j \geq 0$ no matter how large we make x_j . If $\bar{a}_{ji} > 0$, then increasing x_j would make $\bar{b}_i - \bar{a}_{ji} x_j$ to become smaller and eventually hit zero. In order to ensure that all variable are non-negative, we cannot increase x_j beyond a certain point.

For each i ($i=1, \dots, m$) such that $\bar{a}_{ji} > 0$, the value of x_j that will make x_{B_i} go to 0 can be found by observing

$$x_{B_i} = \bar{b}_i - \bar{a}_{ji} x_j$$

$$\text{if } x_{B_i} = 0 \Rightarrow \bar{b}_i - \bar{a}_{ji} x_j = 0 \Rightarrow x_j = \frac{\bar{b}_i}{\bar{a}_{ji}}$$

Thus, the largest possible value that we can assign x_j and ensure that all basic variables are positive is

$$\min \left\{ \frac{\bar{b}_i}{\bar{a}_{ji}} : i=1, \dots, m, \bar{a}_{ji} > 0 \right\} \quad (1)$$

Expression (1) is called the minimum ratio test. We are interested in which index this min is attained.

Suppose we find from min ratio test that $x_j = \frac{\bar{b}_k}{\bar{a}_{jk}}$. Then variable x_j (which

is non-basic) becomes basic variable and variable x_{B_k} becomes non-basic. All

other basic variable remain basic variable (positive). In executing this procedure (of exchanging basic variable with non-basic) we move from one extreme pt of X to another extreme pt.

Thm If reduced cost $\bar{z}_j - \bar{c}_j \geq 0$ for all $j \in J$, then the current basic feasible solution is optimal.

□

We know if LP problem has an optimal solution, it occurs at an extreme pt. There exists one-to-one correspondence between extreme pts and basic feasible solutions. If $\bar{z}_j - \bar{c}_j \geq 0$ for all $j \in J$, then $\frac{\partial z}{\partial x_j} \leq 0$.

Thus, we cannot increase value of the objective function z by increasing any of x_j from 0 to some positive value. Since moving from one extreme pt to another will not improve value of z , we conclude that solution is optimal. ■

Thm In a maximization problem, if $\bar{a}_{ij} \leq 0$ for all $i=1, \dots, m$, and reduced cost $\bar{z}_j - \bar{c}_j < 0$, then LP problem is unbounded.

□

$\bar{z}_j - \bar{c}_j < 0$ implies $\frac{\partial z}{\partial x_j} > 0 \rightarrow z$ will increase if x_j is increased. Since $\bar{a}_{ij} \leq 0$, we can increase x_j as much as we want and still have basic feasible solution, i.e. $x_k > 0$ (basic variables will never become zero). Since we can keep increasing z , the problem is unbounded. ■

3/6/2013

Remark When we swap non-basic variable x_j with basic variable x_{B_k} , this is equivalent to swapping columns $\begin{matrix} \\ \nearrow \\ A \end{matrix}$ that correspond to variables x_j and x_{B_k} . We need to make sure that the resulting matrix B has m linearly independent columns.

Recall

Lemma Let $\{x_1, \dots, x_{m+1}\}$ be a linearly dependent set of vectors in \mathbb{R}^n and let $\{x_1, \dots, x_m\}$ be a linearly independent set. Assume that $x_{m+1} \neq 0$. Assume that $\alpha_1, \alpha_2, \dots, \alpha_{m+1}$ are set of scalars, not all zero, so that

$$\sum_{i=1}^{m+1} \alpha_i x_i = 0 \Leftrightarrow \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_m x_m + \alpha_{m+1} x_{m+1} = 0$$

For any $j \in \{1, \dots, m\}$ such that $\alpha_j \neq 0$, if we replace x_j with x_{m+1} , then this new set of vectors is linearly independent.

We need to write the column of matrix A that corresponds to variable x_j as a linear combination of columns of matrix B :

$$(1) \quad \alpha_1 b_1 + \dots + \alpha_m b_m = A_{\cdot j}$$

If we exchange variable x_j with variable, say, x_i , we need to make sure $d_i \neq 0$.

Q How to find these coefficients d_1, \dots, d_m ?

We denoted $\bar{a}_j = B^{-1} A_{\cdot j} \Rightarrow B \bar{a}_j = A_{\cdot j}$
 such we can write as a sum of products of columns of B and components of \bar{a}_j :
components of \bar{a}_j

$$(2) \quad A_{\cdot j} = B_{\cdot 1} \cdot \bar{a}_{j1} + B_{\cdot 2} \cdot \bar{a}_{j2} + \dots + B_{\cdot m} \cdot \bar{a}_{jm}$$

$\underbrace{\bar{a}_{j1}}_{\substack{\text{1st column} \\ \text{of } B}}$ $\underbrace{\bar{a}_{j2}}_{\substack{\text{2nd column} \\ \text{of } B}}$ \dots $\underbrace{\bar{a}_{jm}}_{\substack{\text{m-th column} \\ \text{of } B}}$

$$\bar{a}_j = \begin{bmatrix} \bar{a}_{j1} \\ \bar{a}_{j2} \\ \vdots \\ \bar{a}_{jm} \end{bmatrix}$$

which shows how to write $A_{\cdot j}$ as a linear combination of columns of B . Compare (2) with (1).

Ex Consider the Toy Maker Problem.

The LP problem is

$$\max Z(x_1, x_2) = 7x_1 + 6x_2$$

$$\text{s.t. } 3x_1 + x_2 \leq 120$$

$$x_1 + 2x_2 \leq 160$$

$$x_1 \leq 35$$

$$x_1 \geq 0, x_2 \geq 0$$

We convert this problem into the standard form by introducing slack variables s_1, s_2, s_3 :

$$\text{max } Z(x_1, x_2) = 7x_1 + 6x_2$$

$$\text{s.t. } 3x_1 + x_2 + s_1 = 120$$

$$x_1 + 2x_2 + s_2 = 160$$

$$x_1 + s_3 = 35$$

$$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$$

This yields

$$C = \begin{bmatrix} 7 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} \quad Ax = b$$

$$A = \begin{bmatrix} 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 120 \\ 160 \\ 35 \end{bmatrix}$$

We can begin with matrices

$$\text{basic matrix } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}$$

non-basic matrix $\uparrow \quad \uparrow$

$A_{\cdot 1} \quad A_{\cdot 2}$

$$X_B = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix}, \quad X_N = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$C_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_N = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

and

$$B^{-1} b = \begin{bmatrix} 120 \\ 160 \\ 35 \end{bmatrix}, \quad B^{-1} N = \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$\text{(since } B \text{ is unit matrix } B = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore,

cost (objective function \geq when $X_N = 0$)

$$C_B^T B^{-1} b = 0, \quad C_B^T B^{-1} N = [0 \ 0]$$

$$\text{Reduced cost: } C_B^T B^{-1} N - C_N = [-7 \ -6]$$

Using this information, we can compute

$$x_1 \quad C_B^T B^{-1} A_{\cdot 1} - C_1 = -7 < 0$$

$$x_2 \quad C_B^T B^{-1} A_{\cdot 2} - C_2 = -6 < 0$$

Based on this information, we can choose either x_1 or x_2 to become basic variable and then the value of the objective function would increase. Let's choose x_1 to enter the basis. Then we need to decide which

basic variable should leave the basis.
 We must investigate the elements of $B^{-1}A_{\cdot 1}$ and the current basic solution $B^{-1}\bar{b}$. We need to perform the minimum ratio test.

$B^{-1}A_{\cdot 1}$ is the 1st column of $B^{-1}N$, i.e.

$$B^{-1}A_{\cdot 1} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \bar{a}_1$$

$$B^{-1}\bar{b} = \begin{bmatrix} 120 \\ 160 \\ 35 \end{bmatrix} = \bar{b} : \text{basic solution}$$

Performing min ratio test, we get

$$\min \left\{ \frac{120}{3}, \frac{160}{1}, \frac{35}{1} \right\}$$

$$S_1 \quad S_2 \quad S_3$$

In this case, the min value $\frac{35}{1} = 35$ is achieved on variable S_3 . This implies that variable x_1 enters the basis and variable S_3 leaves the basis. The new basic and non-basic variables become:

$$x_B = \begin{bmatrix} S_1 \\ S_2 \\ x_1 \end{bmatrix}$$

$$x_N = \begin{bmatrix} S_3 \\ x_2 \end{bmatrix}$$

$$c_B = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}, c_N = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

and matrices become:

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

We simply swapped the column corresponding to variable x_1 with column corresponding to s_3 .

3/8/2013

Simplex Method. Algebraic Form

Toy Maker Problem (Cont'd)

After 1st round of operations, we have

$$x_B = \begin{bmatrix} s_1 \\ s_2 \\ x_1 \end{bmatrix} \quad x_N = \begin{bmatrix} s_3 \\ x_2 \end{bmatrix}$$

$$c_B = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} \quad c_N = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

and matrices B and N

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

With these new matrices, we compute

$$\underbrace{B^{-1} b}_{\text{basic solution}} = \begin{bmatrix} 15 \\ 125 \\ 35 \end{bmatrix} \quad B^{-1} N = \begin{bmatrix} -3 & 1 \\ -1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s_3 \\ x_2 \end{bmatrix}$$

The cost information becomes

$$\underbrace{c_B^T B^{-1} b}_{\text{value of objective function at current basic solution}} = 245$$

value of objective
function at current
basic solution

$$C_B^T B^{-1} N = [7 \ 0]$$

$$C_B^T B^{-1} N - C_N = [7 \ -6]$$

reduced cost

s_3

x_2

\uparrow

\uparrow

Using this information we can compute

$$s_3 \quad C_B^T B^{-1} A_{\cdot 3} - C_3 = 7 > 0$$

$$x_2 \quad C_B^T B^{-1} A_{\cdot 2} - C_2 = -6 < 0 \Rightarrow \frac{\partial z}{\partial x_2} > 0$$

Hence, variable x_2 has to enter the basis.

To decide which variable has to leave the basis, we use min ratio test.

We know that $B^{-1} A_{\cdot 2}$ is the 2nd column of $B^{-1} N$:

$$B^{-1} A_{\cdot 2} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \bar{a}_2$$

$$B^{-1} b = \bar{b} = \begin{bmatrix} 15 \\ 125 \\ 35 \end{bmatrix}$$

$$\text{min. ratio test: } \min \left\{ \underbrace{\frac{15}{1}}_{S_1}, \underbrace{\frac{125}{2}}_{S_2} \right\} = 15$$

The min ratio occurs on variable $s_1 \Rightarrow s_1$ has to leave the basis, while x_2 enters the basis. \Rightarrow swap variables s_1 and x_2 and corresponding columns of A

$$x_B = \begin{bmatrix} x_2 \\ s_2 \\ x_1 \end{bmatrix}$$

$$x_N = \begin{bmatrix} s_3 \\ s_1 \end{bmatrix}$$

$$c_B = \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix}$$

$$c_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

B and N
and matrices become

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

↑ ↑ ↑ ↑ ↑
x₂ s₂ x₁ s₃ s₁

The derived matrices are then

$$B^{-1} b = \begin{bmatrix} 15 \\ 95 \\ 35 \end{bmatrix} \leftarrow x_2$$

basic solution

$$B^{-1} N = \begin{bmatrix} -3 & 1 \\ 5 & -2 \\ 1 & 0 \end{bmatrix}$$

↑ ↑
s₃ s₁

The cost information becomes

$$C_B^T B^{-1} b = 335, \quad C_B^T B^{-1} N = [-11 \quad 6]$$

$$C_B^T B^{-1} N - C_N = [-11 \quad 6]$$

reduced cost ↑ ↑
s₃ s₁
col. 5 col. 3
of A of A

$$s_3 \quad C_B^T B^{-1} A_{:,5} - c_5 = -11 < 0 \Rightarrow \frac{\partial z}{\partial s_3} > 0$$

$$s_1 \quad C_B^T B^{-1} A_{:,3} - c_3 = 3 > 0$$

Hence, variable s₃ has to enter the basis.

To decide which variable leaves the basis
 we use min ratio test using $B^{-1}A_{\cdot 5}$ which
 is 1st column of $B^{-1}N$

$$B^{-1}A_{\cdot 5} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix} = \bar{a}_5$$

$$B^{-1}\bar{b} = \bar{b} = \begin{bmatrix} 15 \\ 95 \\ 35 \end{bmatrix}$$

Min ratio test : $\left\{ \dots, \frac{95}{5}, \frac{35}{1} \right\} = \frac{95}{5}$

↑
do not use ↑
since $-3 < 0$

$x_2 \quad s_2 \quad x_1$

Min $= \frac{95}{5}$ occurs at variable s_2 , hence variable
 s_2 leaves the basis, i.e. we swap s_2 with s_3 .

$$x_B = \begin{bmatrix} x_2 \\ s_3 \\ x_1 \end{bmatrix}$$

$$x_N = \begin{bmatrix} s_2 \\ s_1 \end{bmatrix}$$

$$c_B = \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix}$$

$$c_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and matrices B and N become

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The derived matrices

$$B^{-1}\bar{b} = \begin{bmatrix} 72 \\ 19 \\ 16 \end{bmatrix} \leftarrow \begin{array}{l} x_2 \\ s_3 \\ x_1 \end{array}$$

Basic solution

$$B^{-1}N = \begin{bmatrix} 6/10 & -1/5 \\ 1/5 & -2/5 \\ -1/5 & 2/5 \end{bmatrix}$$

The cost information becomes

$$\underbrace{C_B^T B^{-1} b}_{\text{value of } z} = 544 \quad C_B^T B^{-1} N = \begin{bmatrix} 11/5 & 8/5 \\ - & - \end{bmatrix}$$
$$\underbrace{C_B^T B^{-1} N - C_N}_{\text{reduced cost}} = \begin{bmatrix} 11/5 & 8/5 \\ - & - \end{bmatrix} \quad \begin{matrix} \uparrow & \uparrow \\ s_2 & s_1 \end{matrix}$$

Since all entries of reduced cost vector are positive (not negative), this implies that

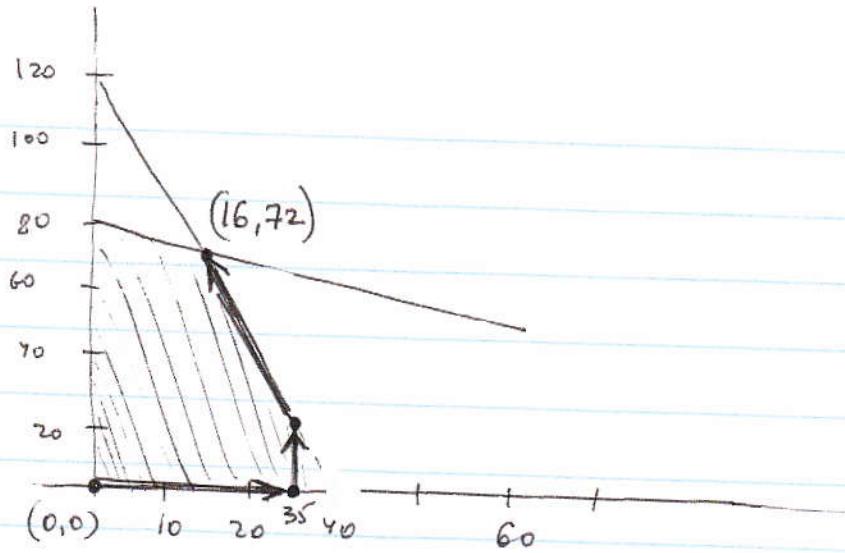
$$\frac{\partial z}{\partial s_2} < 0, \quad \frac{\partial z}{\partial s_1} < 0 \quad \text{and we can't make}$$

value of z bigger by changing s_2 or s_1 from 0 to some positive value. Hence, our solution is optimal.

The final solution is

$$X_B^* = \begin{bmatrix} X_2 \\ s_3 \\ X_1 \end{bmatrix} = \begin{bmatrix} 72 \\ 19 \\ 16 \end{bmatrix} = B^{-1} b$$

Hence, $X_1 = 16$ and $X_2 = 72$ with $z^* = 544$.
The path of extreme points we actually took is depicted below.



$$(x_1, x_2) = (0, 0)$$

↓

$$(x_1, x_2) = (35, 0)$$

↓

$$(x_1, x_2) = (35, 15)$$

↓

$$(x_1, x_2) = (16, 72)$$

$x_1 \geq 0, x_2 \geq 3$

3/18/2013

$$\begin{array}{c} \text{---} \\ | \quad | \quad | \quad | \\ 0 \quad 3 \quad | \quad | \quad | \end{array}$$

Simplex Method - Tableau Form

Performing simplex method in algebraic form is not very efficient from the storage perspective. A more compact form is so called tableau form that is a bigger size matrix that has all necessary information.

To derive simplex Tableau, consider a LP in standard form:

$$\left\{ \begin{array}{l} \max \quad c^T x \\ \text{s.t.} \quad Ax = b \\ \quad \quad \quad x \geq 0 \end{array} \right.$$

and separate A into
basic and non-basic parts

Introduce variable z
to problem as

\checkmark and rewrite

$$(0) \quad \begin{aligned} \max \quad & z \\ \text{s.t.} \quad & z - c_B^T x_B - c_N^T x_N = 0 \\ & B x_B + N x_N = b \end{aligned}$$

$$x_B, x_N \geq 0$$

Multiply both sides of (1) by B^{-1} :

(2)

$$x_B + B^{-1} N x_N = B^{-1} b \quad | \cdot C_B^T$$

Multiply both sides of (2) by C_B^T :

(3)

$$C_B^T x_B + C_B^T B^{-1} N x_N = C_B^T B^{-1} b$$

and add equations (2) and (3):

$$z + 0^T x_B + C_B^T B^{-1} N x_N - C_N^T x_N = C_B^T B^{-1} b$$

0^T : zero vector of appropriate size

Then we can rewrite the last equation as

(4)

$$z + 0^T x_B + (C_B^T B^{-1} N - C_N^T) x_N = C_B^T B^{-1} b$$

We can now write equations (4) and (2)
as a large matrix (tableau):

$A \in \mathbb{R}^{m \times n}$

	\bar{z}	x_B	x_N	RHS	
\bar{z}	1	0	$C_B^T B^{-1} N - C_N^T$	$C_B^T B^{-1} b$	row 0
x_B	0	I	$B^{-1} N$	$B^{-1} b$	rows 1 through m

$(m+1) \times (n+1)$
matrix

need to use basic solution
for MRT (min ratio test)

reduced cost
current value
of objective
function