

3/20/2013

Simplex Method - Tableau Form (Cont'd)

(*)

| | Z | X_B | X_N | RHS |
|-------|---|-------|--------------------------|------------------|
| Z | 1 | 0 | $C_B^T B^{-1} N - C_N^T$ | $C_B^T B^{-1} b$ |
| X_B | 0 | I | $B^{-1} N$ | $B^{-1} b$ |

Note The first row of $(m+1) \times (m+1)$ matrix contains reduced cost $C_B^T B^{-1} N - C_N^T$ and the current value of the objective function (cost): $C_B^T B^{-1} b$. The remainder consists of identity I matrix of size $m \times m$, matrix $B^{-1} N$ and basic solution $B^{-1} b$.

This matrix representation (or tableau representation) contains all information we need to execute the simplex algorithm. An entering variable is chosen from the columns containing the reduced cost and matrix $B^{-1} N$. Naturally, a column with a negative reduced cost is chosen. We then choose a leaving variable by performing the minimum ratio test (MRT) on the chosen column (from $B^{-1} N$) and the right-hand side (RHS) column ($B^{-1} b$). We pivot on the element at the entering column and leaving row and this

transforms the tableau into a new tableau that represents the new basic feasible solution.

ex We consider again the Toy Maker problem. We want to implement the simplex method in tableau form. The standard form is

$$\begin{aligned} \max z(x_1, x_2) &= 7x_1 + 6x_2 \\ \text{s.t.} \quad 3x_1 + x_2 + s_1 &= 120 \\ x_1 + 2x_2 + s_2 &= 160 \\ x_1 + s_3 &= 35 \\ x_1, x_2, s_1, s_2, s_3 &\geq 0. \end{aligned}$$

Let s_1, s_2, s_3 be basic variables and x_1, x_2 : nonbasic variables. With this choice we have

$$x_B = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \quad x_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

then

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$c_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad c_N = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

see notes
from
3/6/2013

Reduced matrices:

$$B^{-1}b = \begin{bmatrix} 120 \\ 160 \\ 35 \end{bmatrix}$$

$$B^{-1}N = \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$C_B^T B^{-1}b = 0$$

value of
object. f^*

$$C_B^T B^{-1}N - C_N^T = [-7 \quad -6]$$

reduced cost

Thus, our tableau is:

| | | X_N | | X_B | | | |
|-------|-----|-------|-------|-------|-------|-------|-----|
| | Z | x_1 | x_2 | s_1 | s_2 | s_3 | RHS |
| Z | 1 | -7 | -6 | 0 | 0 | 0 | 0 |
| s_1 | 0 | 3 | 1 | 1 | 0 | 0 | 120 |
| s_2 | 0 | 1 | 2 | 0 | 1 | 0 | 160 |
| s_3 | 0 | 1 | 0 | 0 | 0 | 1 | 35 |

Annotations:

- reduced cost (points to -7 and -6)
- object. function value $C_B^T B^{-1}b$ (points to 0)
- $B^{-1}N$ (points to columns 2 and 3)
- I (points to columns 4, 5, and 6)
- $B^{-1}b$ (points to column 7)

Noticed that columns were swapped comparing to representation in (*). This is because we wrote x as $[x_1, x_2, s_1, s_2, s_3]^T$.

Using this information, we see that either variables x_1 or x_2 can enter (since reduced cost components are both negative). Let x_2 enter the basis.

| | z | x_1 | x_2 | s_1 | s_2 | s_3 | RHS | MRT(x_2) |
|-------|-------|-------|-------|-------|-------|-------|-----|-----------------------|
| row 0 | z | 1 | -7 | -6 | 0 | 0 | 0 | |
| row 1 | s_1 | 0 | 3 | 1 | 1 | 0 | 120 | $120 = \frac{120}{1}$ |
| row 2 | s_2 | 0 | 1 | 2 | 0 | 1 | 160 | $80 = \frac{160}{2}$ |
| row 3 | s_3 | 0 | 1 | 0 | 0 | 1 | 35 | - |

$B^{-1}A_{.2} = \bar{a}_2$ $B^{-1}b = \bar{b}$

To decide which variable has to leave, we do minimum ratio test (MRT) on the column that contains x_2 and RHS column. We can write ratios from MRT next to RHS column. MRT will tell us which variable has to leave. Then we put a box around the element on which we will pivot.

Min is achieved on variable s_2 (row 2). We will pivot on element 2. We will multiply row 2 by $\frac{1}{2}$ (to that element $2 \rightarrow 1$) and eliminate variable x_2 from rows above and below row 2. After that column corresponding to x_2 will become

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} z \\ S_1 \\ S_2 \\ S_3 \end{array} \left[\begin{array}{c|ccccccc} 1 & -7 & -6 & 0 & 0 & 0 & 0 \\ \hline 0 & 3 & 1 & 1 & 0 & 0 & 120 \\ 0 & 1 & \boxed{2} & 0 & 1 & 0 & 160 \\ 0 & 1 & 0 & 0 & 0 & 1 & 35 \end{array} \right] \cdot \frac{1}{2}$$

$$\begin{array}{l} \text{row 0} \\ \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{array} \begin{array}{l} z \\ S_1 \\ S_2 \\ S_3 \end{array} \left[\begin{array}{c|ccccccc} 1 & -7 & -6 & 0 & 0 & 0 & 0 \\ \hline 0 & 3 & 1 & 1 & 0 & 0 & 120 \\ 0 & 1/2 & \boxed{1} & 0 & 1/2 & 0 & 80 \\ 0 & 1 & 0 & 0 & 0 & 1 & 35 \end{array} \right]$$

row 0 \leftarrow row 0 + 6 \cdot row 2

row 1 \leftarrow row 1 - row 2

row 3 stays as it is

$$\begin{array}{l} z \\ S_1 \\ X_2 \\ S_3 \end{array} \left[\begin{array}{c|ccccccc} 1 & -7 + \frac{6}{2} = -4 & \boxed{0} & 0 & 0 & \frac{1}{2} \cdot 6 = 3 & 0 & 80 + 6 = 480 \\ \hline 0 & 3 - \frac{1}{2} = 2.5 & \boxed{0} & 0 & 1 & -\frac{1}{2} = -0.5 & 0 & 120 - 80 = 40 \\ 0 & 1/2 & \boxed{1} & 0 & 0 & 1/2 & 0 & 80 \\ 0 & 1 & \boxed{0} & 0 & 0 & 0 & 1 & 35 \end{array} \right]$$

3/22/2013

Ex Toy Maker problem in Tableau Form:
 Last time we obtained the following
 Updated tableau:

| | z | x_1 | x_2 | s_1 | s_2 | s_3 | RHS |
|-----------------------|-----|-------|-------|-------|-------|-------|-----|
| z | 1 | -4 | 0 | 0 | 3 | 0 | 480 |
| s_1 | 0 | 2.5 | 0 | 1 | -0.5 | 0 | 40 |
| $s_2 \rightarrow x_2$ | 0 | 0.5 | 1 | 0 | 0.5 | 0 | 80 |
| s_3 | 0 | 1 | 0 | 0 | 0 | 1 | 35 |

\swarrow 2nd col. \downarrow 1st col. \searrow 3rd col.
 I

We can see that variable x_1 is a valid entering variable since it has negative reduced cost (-4). We perform again minimum ratio test:

| | z | x_1 | x_2 | s_1 | s_2 | s_3 | RHS | MRT (x_1) |
|-------------------------|-----|-------|-------|-------|-------|-------|-----|----------------------------------|
| row 0 z | 1 | -4 | 0 | 0 | 3 | 0 | 480 | |
| row 1 $\rightarrow s_1$ | 0 | 2.5 | 0 | 1 | -0.5 | 0 | 40 | $\frac{40}{2.5} = 16 \leftarrow$ |
| row 2 x_2 | 0 | 0.5 | 1 | 0 | 0.5 | 0 | 80 | $\frac{80}{0.5} = 160$ |
| row 3 s_3 | 0 | 1 | 0 | 0 | 0 | 1 | 35 | $\frac{35}{1} = 35$ |

MRT tells us that variable s_1 has to leave the basis and we need to pivot on element $\boxed{2.5}$, i.e. eliminate \uparrow basic variable x_1 from row 0 and rows 2 and 3

After pivoting we obtain a new tableau:

| | z | x_1 | x_2 | s_1 | s_2 | s_3 | RHS | |
|-----------------------|-----|-------|-------|-------|-------|-------|-----|------------------|
| z | 1 | 0 | 0 | 1.6 | 2.2 | 0 | 544 | cost |
| $s_1 \rightarrow x_1$ | 0 | 1 | 0 | 0.4 | -0.2 | 0 | 16 | $\leftarrow x_1$ |
| x_2 | 0 | 0 | 1 | -0.2 | 0.6 | 0 | 72 | $\leftarrow x_2$ |
| s_3 | 0 | 0 | 0 | -0.4 | 0.2 | 1 | 19 | |

$B^{-1}b$ basic solution

all the reduced costs for non-basic variables (s_1 and s_2) are positive, so this is the optimal solution of our LP problem:

$$x_1 = 16$$

$$x_2 = 72$$

and

$$z_{opt} = 544$$

This solution agrees with the one we found using algebraic form of simplex method.

Identifying Unboundedness

We already discussed situations when problem is unbounded. We saw a theorem: in a maximization problem, if $\bar{a}_{ij} < 0$ for all $i=1, \dots, m$, and $z_j - c_j < 0$, then the linear programming problem is

elements of $\bar{a}_{ij} = B^{-1}A_{ij}$

unbounded. (see lecture 3/4/2013).

This condition occurs when variable x_j should enter the basis because $\partial z / \partial x_j > 0$ and there is no blocking basis variable. That is, we can arbitrarily increase x_j without causing any variable to become negative.

Ex Consider a LP problem (considered before)

$$\max z(x_1, x_2) = 2x_1 - x_2$$

$$\text{s.t. } x_1 - x_2 \leq 1$$

$$2x_1 + x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

We can convert this problem into standard form by introducing slack variable s_1 and surplus variable s_2 :

$$\max z(x_1, x_2) = 2x_1 - x_2$$

$$\text{s.t. } x_1 - x_2 + s_1 = 1$$

$$2x_1 + x_2 - s_2 = 6$$

$$x_1, x_2, s_1, s_2 \geq 0$$

This gives matrices

$$c = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$x_1=0, x_2=0$ is not a valid basic solution since we have both slack and surplus variable. (since then $s_1=170$ but $s_2=-6 < 0$)

Assume that $s_1=s_2=0$, so we have

$$x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad x_N = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \quad c_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad c_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

This yields:

$$B^{-1}b = \begin{bmatrix} 7/3 \\ 4/3 \end{bmatrix} \quad B^{-1}N = \begin{bmatrix} 1/3 & -1/3 \\ -2/3 & -1/3 \end{bmatrix}$$

$\begin{matrix} s_1 & & s_2 \\ \downarrow & & \downarrow \end{matrix}$

We also have cost information:

$$c_B^T B^{-1}b = \frac{10}{3} \quad c_B^T B^{-1}N - c_N^T = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \end{bmatrix}$$

$\begin{matrix} \uparrow & & \uparrow & \uparrow \\ z \text{ cost} & & \text{reduced cost} & s_1 & s_2 \end{matrix}$

We can put this information in tableau form:

| | | x_1 | x_2 | s_1 | s_2 | RHS |
|-------|---|-------|-------|--------|--------|--------|
| z | 1 | 0 | 0 | $4/3$ | $-1/3$ | $10/3$ |
| x_1 | 0 | 1 | 0 | $1/3$ | $-1/3$ | $7/3$ |
| x_2 | 0 | 0 | 1 | $-2/3$ | $-1/3$ | $4/3$ |

We see that variable s_2 has to enter the

basis since reduced cost is negative ($-\frac{1}{3}$). But the column that corresponds to S_2 is all negative. Therefore, there is no minimum ratio test. We can increase S_2 as much as we like and this will increase the value of the objective function without bounds and none of other variables will become negative (i.e. we will not violate feasibility condition).

In fact, we have shown that the ray with vertex

$$x_0 = \begin{bmatrix} 7/3 \\ 4/3 \\ 0 \\ 0 \end{bmatrix}$$

and direction

$$d = \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ 1 \end{bmatrix}$$

is entirely contained inside the polyhedral set defined by $AX=b$. This can be seen by observing the following.

$$x_B = B^{-1}b - B^{-1}N x_N$$

In our case, we will have $s_2 \neq 0$ (and increasing $s_2 > 0$) and $s_1 = 0$ (will remain zero)

$$x_B = B^{-1}b - B^{-1}A_{\cdot y} \cdot s_2 \quad x_N = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

$$\text{but } B^{-1}b = \begin{bmatrix} 7/3 \\ 4/3 \end{bmatrix} \quad B^{-1}A_{\cdot y} = \begin{bmatrix} -1/3 \\ -1/3 \end{bmatrix}$$

$$\therefore x_B = \begin{bmatrix} 7/3 \\ 4/3 \end{bmatrix} + s_2 \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} : \text{ ray w/ vertex } \begin{bmatrix} 7/3 \\ 4/3 \end{bmatrix} \text{ and direction } \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

We will increase the value of s_2 (s_2 plays a role of a parameter λ in def of a ray) and leave s_1 to be 0. As can be seen from the graph of the feasible region, this ray belongs to the feasible region.

Pt $(\frac{7}{3}, \frac{4}{3})$ is an extreme pt of the feasible region and ray $\begin{bmatrix} 7/3 \\ 4/3 \end{bmatrix} + s_2 \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$ is one of the edges.

Thm In a maximization problem, if $\bar{a}_j \leq 0$ for all $j=1, \dots, m$, and $z_j - c_j < 0$, then the LP problem is unbounded. Furthermore, if \bar{a}_j is the j^{th} column of $B^{-1}N$ (that corresponds to variable x_j) and e_k be the standard unit basis

vector in $\mathbb{R}^{m \times (n-m)}$ where k corresponds to the position of j in matrix N . Then the direction:

$$d = \begin{bmatrix} -a_j \\ e_k \end{bmatrix}$$

is an extreme direction of the feasible region $X = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$

Ex $\bar{a}_j = B^{-1}A_{\cdot j} = \begin{bmatrix} -1/3 \\ -1/3 \end{bmatrix}$. This is the 2nd column of matrix $B^{-1}N$

$$\therefore d = \begin{bmatrix} -(-1/3) \\ -(-1/3) \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ 1 \end{bmatrix} \Rightarrow k=2 \Rightarrow e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Identifying Alternative Optimal Solutions

We saw that if the reduced cost has all positive components, we have found an optimal solution. This happens when $\bar{z}_j - c_j > 0$ for all $j \in J$ (indices of non-basic variables). When $\bar{z}_j - c_j = 0$, the situation is slightly different.

Thm (Identifying alternative optimal solutions, Consider LP problem

$$P \begin{cases} \max & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{cases}$$

For problem P for a given set of non-basic variables with indices $j \in J$, if $z_j - c_j \geq 0$ for all $j \in J$, then the current basic solution is optimal. Further, if $z_j - c_j = 0$ for at least one $j \in J$, then there are alternative optimal solutions. Furthermore, let \bar{a}_j be the j^{th} column of $B^{-1}N$, i.e. $\bar{a}_j = B^{-1}A_{\cdot j}$. Then solutions of problem P are:

$$\begin{cases} x_B = B^{-1}b - \bar{a}_j x_j \\ x_j \in [0, \min \left\{ \frac{\bar{b}_i}{\bar{a}_{ji}} : i=1, \dots, m, \bar{a}_{ji} > 0 \right\}] \\ x_r = 0, \quad \forall r \in J, \quad r \neq j \end{cases}$$

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Alternative optimal solutions

(*) Recall Thm. If $z_j - c_j \geq 0$ for all $j \in J$ (indices of all non-basic variables), then the current basic feasible solution is optimal.

We proved this Thm earlier.

Back to the Thm about identifying alternative optimal solutions that we wrote last time. From Thm (*) it follows that if $z_j - c_j \geq 0$ for all non-basic variables, then $\frac{\partial z}{\partial x_j} \leq 0$ and increasing variable x_j will not increase the value of the objective function. If the reduced cost $z_j - c_j = 0$ for at least one index j , then $\frac{\partial z}{\partial x_j} = 0$ and we can increase x_j from 0 to some positive value (given by MRT) so that the value of the objective function will not change at all. Since the value of objective function is optimal, it follows that we get a set of values of x_j that give alternative optimal solutions.

Ex Consider again the Toy Maker problem with an adjusted objective function.

$$z(x_1, x_2) = 18x_1 + 6x_2.$$

Let basic variable variables be x_1 , s_2 and x_2 .

$$x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_2 \end{bmatrix} \quad x_N = \begin{bmatrix} s_1 \\ s_3 \end{bmatrix} \quad c_B = \begin{bmatrix} 18 \\ 6 \\ 0 \end{bmatrix} \quad c_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The matrices become

$$B = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The derived matrices are

$$B^{-1}b = \begin{bmatrix} 35 \\ 15 \\ 95 \end{bmatrix} \quad B^{-1}N = \begin{bmatrix} 0 & 1 \\ 1 & -3 \\ -2 & 5 \end{bmatrix}$$

$\uparrow \quad \uparrow$
 $s_1 \quad s_3$

The cost information becomes

$$c_B^T B^{-1}b = 720 \quad c_B^T B^{-1}N - c_N^T = \begin{bmatrix} 6 & 0 \\ \uparrow & \uparrow \\ s_1 & s_3 \end{bmatrix}$$

This yields the tableau:

| | z | x_1 | x_2 | s_1 | s_2 | s_3 | RHS |
|-------|-----|-------|-------|-------|-------|-------|-----|
| z | 1 | 0 | 0 | 6 | 0 | 0 | 720 |
| x_1 | 0 | 1 | 0 | 0 | 0 | 1 | 35 |
| x_2 | 0 | 0 | 1 | 1 | 0 | -3 | 15 |
| s_2 | 0 | 0 | 0 | -2 | 1 | 5 | 95 |

The reduced cost for s_3 is 0. This means that if we allow s_3 to enter the basis, the value of the objective function will not change. Performing the MRT on variable s_3 , however, we see that

| | z | x_1 | x_2 | s_1 | s_2 | s_3 | RHS | MRT(s_3) |
|-------|-----|-------|-------|-------|-------|-------|-----|-------------------------|
| z | 1 | 0 | 0 | 6 | 0 | 0 | 720 | |
| x_1 | 0 | 1 | 0 | 0 | 0 | 1 | 35 | $35/1 = 35$ |
| x_2 | 0 | 0 | 1 | 1 | 0 | -3 | 15 | |
| s_2 | 0 | 0 | 0 | -2 | 1 | 5 | 95 | $\rightarrow 95/5 = 19$ |

$s_3 \in [0, 19]$

s_2 will have to leave the basis while s_3 will enter the basis and will vary on $s_3 \in [0, 19]$.

Therefore, any solution of the form

$$(2) \begin{bmatrix} x_1 \\ x_2 \\ s_2 \end{bmatrix} = \begin{bmatrix} 35 \\ 15 \\ 95 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} s_3, \quad s_3 \in [0, 19]$$

$\begin{matrix} \swarrow x_i \\ \text{B}^{-1}b \\ \text{B}^{-1}A_{.5} \end{matrix}$

is an optimal alternative solution:

$$x_B = B^{-1}b - B^{-1}N x_N$$

only $x_j \neq 0$, $j \in \mathcal{J} \Rightarrow$

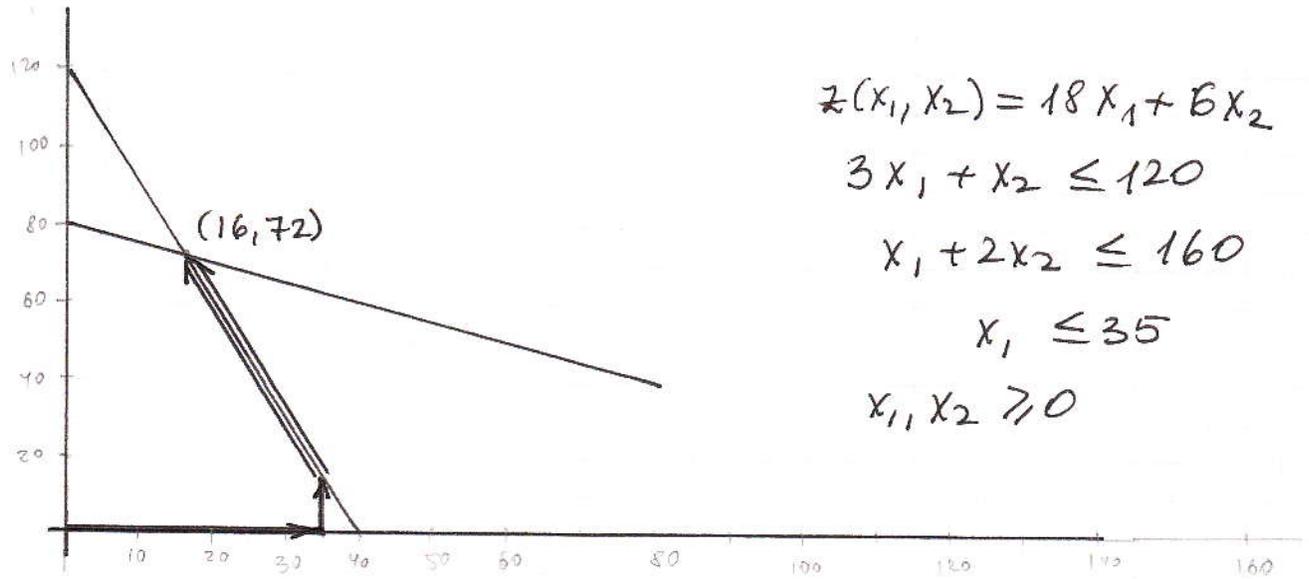
$$x_B = B^{-1}b - B^{-1}A_{\cdot j} x_j$$

$$B^{-1}b = \begin{bmatrix} 35 \\ 15 \\ 95 \end{bmatrix}$$

$$B^{-1}A_{\cdot j} = B^{-1}A_{\cdot 5} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$$

$$x_j = s_3$$

Solution (2) describes the edge shown in the Fig. on the next page.



Degeneracy and Convergence

We will look at the example of degeneracy and its impact on the simplex algorithm.

Ex Consider a modified Toy Maker problem with added extra constraint:

$$\max 7x_1 + 6x_2$$

$$\text{s.t. } 3x_1 + x_2 \leq 120$$

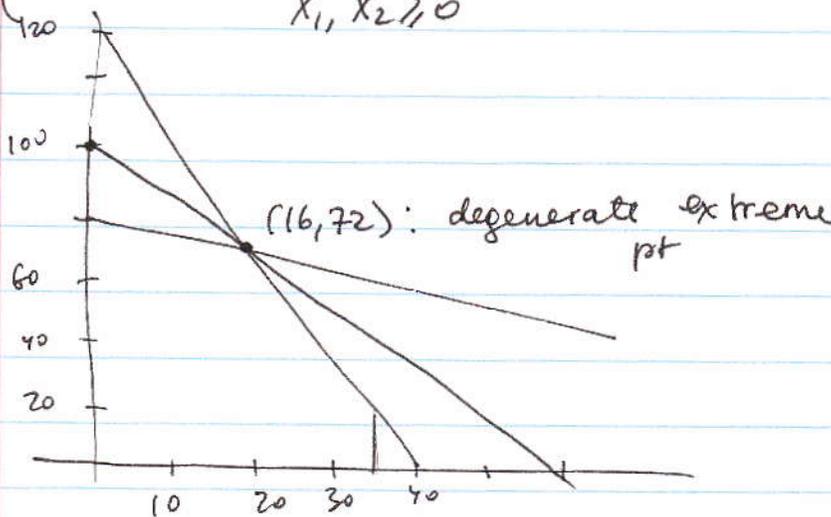
$$x_1 + 2x_2 \leq 160$$

$$x_1 \leq 35$$

$$\frac{7}{4}x_1 + x_2 \leq 100$$

additional constraint

$$x_1, x_2 \geq 0$$



$$\frac{7}{4}x_1 + x_2 = 100$$

$$x_2 = 0 \Rightarrow x_1 = \frac{400}{7} \approx 57.14$$

We bring this problem to the standard form by introducing slack variables

$$\max \quad 7x_1 + 6x_2$$

$$\text{s.t.} \quad 3x_1 + x_2 + s_1 = 120$$

$$x_1 + 2x_2 + s_2 = 160$$

$$x_1 + s_3 = 35$$

$$\frac{7}{4}x_1 + x_2 + s_4 = 100$$

$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

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Degeneracy and Convergence (Cont'd)

Ex Consider a modified Toy Maker problem with extra condition imposed (see previous lecture).

Suppose we start with

$$x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_2 \\ s_4 \end{bmatrix} \quad x_N = \begin{bmatrix} s_1 \\ s_3 \end{bmatrix} \quad (\Rightarrow s_1=0, s_3=0)$$

In this case, the matrices are

$$B = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 7/4 & 1 & 0 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The derived matrices are

$$B^{-1}b = \begin{bmatrix} 35 \\ 15 \\ 95 \\ 95/4 \end{bmatrix} \quad B^{-1}N = \begin{bmatrix} 0 & 1 \\ 1 & -3 \\ -2 & 5 \\ -1 & 5/4 \end{bmatrix}$$

we start with $x_1 = 35, x_2 = 15, s_2 = 95, s_4 = 95/4 = 23.75$

$$c_B^T B^{-1}b = 335$$

cost

$$c_B^T B^{-1}N - c_N^T = \begin{bmatrix} 6 & -11 \end{bmatrix}$$

reduced cost

$\uparrow \quad \uparrow$
 $s_1 \quad s_3$

The tableau representation is

| | z | x_1 | x_2 | s_1 | s_2 | s_3 | s_4 | RHS | MRT(s_3) |
|-------|-----|-------|-------|-------|-------|---------------|-------|----------------|---|
| z | 1 | 0 | 0 | 6 | 0 | -11 | 0 | 335 | |
| x_1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 35 | $\frac{35}{1} = 35$ |
| x_2 | 0 | 0 | 1 | 1 | 0 | -3 | 0 | 15 | |
| s_2 | 0 | 0 | 0 | -2 | 1 | 5 | 0 | 95 | $\frac{95}{5}$ |
| s_4 | 0 | 0 | 0 | -1 | 0 | $\frac{5}{4}$ | 1 | $\frac{95}{4}$ | $\frac{95}{4} \cdot \frac{4}{5} = \frac{95}{5}$ |

From the tableau we see that variable s_3 has to enter the basis (the reduced cost $-11 < 0$). Performing the MRT on variable s_3 , we notice that there is a tie between which variable should leave the basis:

both s_2 and s_4 give the ratio $\frac{95}{5} = \frac{95/5}{4/5} = 19$

This is because when s_3 enters the basis, the next point is a degenerate extreme pt.

Suppose we choose that s_4 leaves the basis. Then we have to pivot around element $\frac{5}{4}$. Our new tableau becomes

| | Z | X_1 | X_2 | S_1 | S_2 | S_3 | S_4 | RHS | MRT(s_i) |
|-----------------------|-----|-------|-------|---|-------|-------|--------|---|-----------------------|
| Z | 1 | 0 | 0 | $-14/5$ | 0 | 0 | $44/5$ | 544 | |
| X_1 | 0 | 1 | 0 | $4/5$ | 0 | 0 | $-4/5$ | 16 | $\frac{16}{4/5} = 20$ |
| X_2 | 0 | 0 | 1 | $-7/5$ | 0 | 0 | $12/5$ | 72 | |
| S_2 | 0 | 0 | 0 | 2 | 1 | 0 | -4 | 0 | $\frac{0}{2} = 0$ |
| $S_4 \rightarrow S_3$ | 0 | 0 | 0 | $-4/5$ | 0 | 1 | $4/5$ | 19 | |

We observe two things:

$B^{-1}b$
basic solution

- One of the basic variables (S_2) is zero, even though it is basic. This is an indicator of degeneracy of an extreme point.
- Reduced cost corresponding to variable S_1 is negative ($-14/5$), so variable S_1 enters the basis.

Performing MRT on S_1 , we find that variable S_2 enters the basis and we need to pivot around element 2. A new tableau form becomes:

| | z | x_1 | x_2 | s_1 | s_2 | s_3 | s_4 | RHS |
|-----------------------|-----|-------|-------|-------|--------|-------|--------|-----|
| z | 1 | 0 | 0 | 0 | $7/5$ | 0 | $16/5$ | 544 |
| x_1 | 0 | 1 | 0 | 0 | $-2/5$ | 0 | $4/5$ | 16 |
| x_2 | 0 | 0 | 1 | 0 | $7/10$ | 0 | $-2/5$ | 72 |
| $s_2 \rightarrow s_1$ | 0 | 0 | 0 | 1 | $1/2$ | 0 | -2 | 0 |
| s_3 | 0 | 0 | 0 | 0 | $2/5$ | 1 | $-1/5$ | 19 |

Notice that the value of the objective function $C_B B^{-1} b = 544$ has not changed, because we really have not moved to a new extreme point. We simply changed from one representation of degenerate point to another. This was expected since the MRT gave value 0 that meant that variable s_1 had to "increase" from 0 to 0, i.e. variable s_1 has not changed, so as the value of the objective function has not changed. The reduced cost has changed - all components are 0 and the simplex method terminates with the optimal solution $x_1 = 16, x_2 = 72$.

Then Consider a linear programming problem P . Let $B \in \mathbb{R}^{m \times m}$ be a basic matrix corresponding to some set of basic variables x_B . Let $\bar{b} = B^{-1} b$. If $\bar{b}_j = 0$ for some $j = 1, \dots, m$, then $x_B = \bar{b}$ and $x_N = 0$ is a degenerate extreme point

of the feasible region of Problem P.

D

At any basic feasible solution we have chosen m variables as basic. This basic feasible solution satisfies $Bx_B = b$ and thus provides m binding constraints. The remaining variables are chosen as non-basic and set to zero, thus $x_N = 0$, which provides $n-m$ binding constraints on the non-negativity constraints (i.e., $x \geq 0$). If there is a basic variable that is zero, then an extra non-negativity constraint is binding at that extreme point. Thus $n+1$ constraints are binding and, by definition, the extreme point must be degenerate.

The Simplex Algorithm and Convergence

3/29/2013

Simplex Algorithm in Algebraic Form

1. Given Problem P in standard form with cost vector c , coefficient matrix A and right hand side b , identify an initial basic feasible solution x_B and x_N by any means. Let J be the set of indices of non-basic variables. If no basic feasible solution can be found, STOP, the problem has no solution.
2. Compute the reduced cost ^{row} vector $c_B^T B^{-1} N - c_N^T$. This vector contains $z_j - c_j$, for $j \in J$.
3. If $z_j - c_j \geq 0$ for all $j \in J$, STOP, the current basic feasible solution is optimal. Otherwise, Goto 4.
4. Choose a non-basic variable x_j with $z_j - c_j < 0$. Select $\bar{a}_j = B^{-1} A_j$ from $B^{-1} N$. If $\bar{a}_j \leq 0$, then the problem is unbounded, STOP. Otherwise Goto 5.

5. Let $\bar{b} = B^{-1}b$. Find the index i solving

$$\min \left\{ \frac{\bar{b}_i}{\bar{a}_{ji}} : i=1, \dots, m \text{ and } \bar{a}_{ji} > 0 \right\}$$

This is the minimum ratio test.

6. Set $x_{B_i} = 0$ and $x_j = \frac{\bar{b}_i}{\bar{a}_{ji}}$.

7. Update \bar{b} and goto step 2.

Note A similar algorithm can be formulated for the simplex method in tableau form.

Thm (Convergence of the Simplex Method)

If the feasible region of Problem P has no degenerate extreme points, then the simplex method will terminate in a finite number of steps with an optimal solution to the linear programming problem.

Sketch of the proof. The value of the objective function improves (increases in the case of a maximization problem) each time we exchange basic and non-basic variables. This is ensured by the fact that the entering variable

has a negative reduced cost. Earlier we proved the result that there is a finite number of extreme points for each polyhedral set. Thus, the process of moving from extreme point to extreme point must terminate in a finite # of steps with the largest possible value of the objective function.

Simplex Initialization

We will discuss the issue of finding an initial basic feasible solution to start the execution of the simplex method.

Consider a linear programming problem:

$$\left\{ \begin{array}{l} \max \quad c^T x \\ \text{s.t.} \quad Ax \leq b \\ \quad \quad x \geq 0 \end{array} \right.$$

We can write this problem in the standard form by introducing slack variables:

$$\left\{ \begin{array}{l} \max \quad c^T x \\ \text{s.t.} \quad Ax + I_m x_s = b \\ \quad \quad x, x_s \geq 0 \end{array} \right.$$

where x_s are slack variables, one for each constraint. If $b \geq 0$, then our initial basic feasible solution can be $x=0$ and $x_s = b$ (that is, our initial basic matrix is I_m). We have also explored small size problem where graphical technique can be used to identify an initial basic feasible solution by identifying extreme points of the polyhedral set.

Q How do we identify an initial basic feasible solution in general?

Assume we have a linear programming problem in the standard form:

$$\begin{cases} \max c^T x \\ \text{s.t. } Ax = b \\ x \geq 0 \end{cases}$$

Assume that $b \geq 0$. Each constraint can be written as $A_i \cdot x = b_i$. Let us associate an artificial variable x_{a_i} with each of these constraints and rewrite the constraint as

$$(1) \quad A_i \cdot x + x_{a_i} = b_i$$

Since $b_i \geq 0$, we will require $x_{a_i} \geq 0$. When

$x_{a_i} = 0$, we simply get an original constraint. Thus, if we can find values for the original decision variables x such that $x_{a_i} = 0$, then the constraint i is satisfied. If we can identify values for x such that all artificial variables are zero and m variables of x are non-zero, then the modified constraint (i) will be satisfied as well as the original i constraint, and we have identified an initial basic feasible solution.

Obviously, we want to penalize non-zero artificial variables. This can be done by writing a new linear programming problem:

$$P_1 \left\{ \begin{array}{l} \min e^T x_a \\ \text{s.t. } Ax + I_m x_a = b \\ x, x_a \geq 0 \end{array} \right.$$

where $e^T = (1, 1, \dots, 1)$, $x_a = \begin{pmatrix} x_{a1} \\ x_{a2} \\ \vdots \\ x_{am} \end{pmatrix}$

Note We can see that artificial variables are similar to slack variables, but they should be zero since they do not have any true meaning to the original problem P and we do not want to change the original problem. They are introduced artificially to help us find our initial basic feasible solution.

Ex Consider the following problem:

$$\begin{aligned} \min \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \geq 12 \\ & 2x_1 + 3x_2 \geq 20 \\ & x_1, x_2 \geq 0 \end{aligned}$$

We convert this problem to the standard form by introducing surplus variables:

$$\begin{aligned} \min \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 - s_1 = 12 \\ & 2x_1 + 3x_2 - s_2 = 20 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

It's not clear what a good ^{initial} basic feasible solution should be. Clearly, we cannot set $x_1 = x_2 = 0$, since then we get $s_1 = -12 < 0$, $s_2 = -20 < 0$, which is not feasible.

We will introduce two artificial variables x_{a1} and x_{a2} to create a new problem P_1 :

P_1 {

$$\begin{aligned} \min \quad & x_{a1} + x_{a2} \\ \text{s.t.} \quad & x_1 + 2x_2 - s_1 + x_{a1} = 12 \\ & 2x_1 + 3x_2 - s_2 + x_{a2} = 20 \\ & x_1, x_2, s_1, s_2, x_{a1}, x_{a2} \geq 0 \end{aligned}$$

A basic feasible solution of our artificial problem is $x_{a1} = 12, x_{a2} = 20$. (and $x_1 = x_2 = s_1 = s_2 = 0$)
The matrices are:

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & -1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 12 \\ 20 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_{a1} \\ x_{a2} \end{bmatrix} \quad x_N = \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix}$$

$$c_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad c_N = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & 0 & -1 \end{bmatrix}$$

$$c_B^T B^{-1} b = 32 \quad c_B^T B^{-1} N - c_N^T = \begin{bmatrix} 3 & 5 & -1 & -1 \end{bmatrix}$$

cost (value of objective $\neq 0$) reduced cost \uparrow \uparrow \uparrow \uparrow
 x_1 x_2 s_1 s_2

We can construct an initial tableau form for this problem:

| | z | x_1 | x_2 | s_1 | s_2 | x_{a1} | x_{a2} | RHS |
|----------|-----|-------|-------|-------|-------|----------|----------|-----|
| z | 1 | 3 | 5 | -1 | -1 | 0 | 0 | 32 |
| x_{a1} | 0 | 1 | 2 | -1 | 0 | 1 | 0 | 12 |
| x_{a2} | 0 | 2 | 3 | 0 | -1 | 0 | 1 | 20 |

$B^{-1}b$

$B^{-1}N$

This is a minimization problem, so if $z_j - c_j > 0$, then entering variable x_j will improve (decrease) the value of the objective function. In this case, we could choose either x_1 or x_2 . Let's take x_1 , i.e. variable x_1 will enter the basis.

| | z | x_1 | x_2 | s_1 | s_2 | x_{a1} | x_{a2} | RHS | MRT(x_1) |
|----------------------|-----|-------|-------|-------|-------|----------|----------|-----|--------------------------------|
| z | 1 | 3 | 5 | -1 | -1 | 0 | 0 | 32 | |
| x_{a1} | 0 | 1 | 2 | -1 | 0 | 1 | 0 | 12 | $\frac{12}{1}$ |
| $\rightarrow x_{a2}$ | 0 | 2 | 3 | 0 | -1 | 0 | 1 | 20 | $\frac{20}{2} = 10 \leftarrow$ |

Thus, x_{a2} leaves the basis and x_1 enters the basis. The new tableau becomes:

| | z | x_1 | x_2 | s_1 | s_2 | x_{a1} | x_{a2} | RHS |
|----------|-----|-------|---------------|-------|----------------|----------|----------------|-----|
| z | 1 | 0 | $\frac{1}{2}$ | -1 | $\frac{1}{2}$ | 0 | $-\frac{3}{2}$ | 2 |
| x_{a1} | 0 | 0 | $\frac{1}{2}$ | -1 | $\frac{1}{2}$ | 1 | $-\frac{1}{2}$ | 2 |
| x_1 | 0 | 1 | $\frac{3}{2}$ | 0 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 10 |

In this case, variable x_2 should enter the basis. Performing the MRT we obtain:

$$\begin{array}{c} \downarrow \\ \begin{array}{l} z \\ x_{a_1} \\ x_1 \end{array} \left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & s_1 & s_2 & x_{a_1} & x_{a_2} & \text{RHS} \\ \hline 1 & 0 & 1/2 & -1 & 1/2 & 0 & -3/2 & 2 \\ \hline 0 & 0 & 1/2 & -1 & 1/2 & 1 & -1/2 & 2 \\ \hline 0 & 1 & 3/2 & 0 & -1/2 & 0 & 1/2 & 10 \end{array} \right] \begin{array}{l} \text{MRT}(x_2) \\ \frac{2}{1/2} = 4 \leftarrow \\ \frac{10}{3/2} = \frac{20}{3} \end{array} \end{array}$$

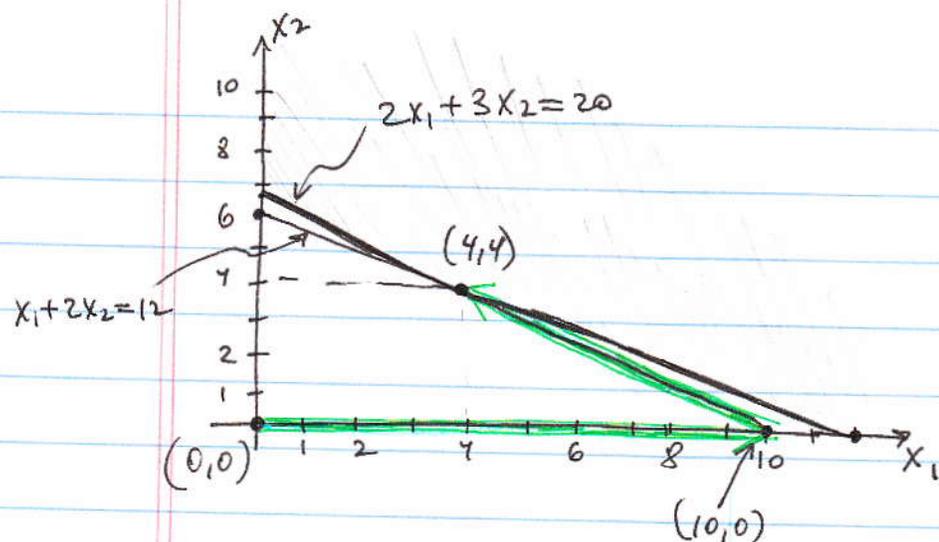
Thus, variable x_{a_1} leaves the basis and we obtain

$$\begin{array}{c} \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \begin{array}{l} z \\ x_2 \\ x_1 \end{array} \left[\begin{array}{c|cccccc|c} z & x_1 & x_2 & s_1 & s_2 & x_{a_1} & x_{a_2} & \text{RHS} \\ \hline 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ \hline 0 & 0 & 1 & -2 & 1 & 2 & -1 & 4 \\ \hline 0 & 1 & 0 & 3 & -2 & -3 & 2 & 4 \end{array} \right] \end{array}$$

$B^{-1}N$ of original problem $\leftarrow B^{-1}$ of original prob

At this point, we have eliminated both artificial variables from the basis and identified an initial basic feasible solution to the original problem:

$$x_1 = 4, \quad x_2 = 4, \quad s_1 = 0, \quad s_2 = 0$$



$$\begin{aligned} \min \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \geq 12 \\ & 2x_1 + 3x_2 \geq 20 \end{aligned}$$

Green arrows show the process of moving to a feasible solution of the original problem.

Now we can continue to solve the original problem. At this point, variables x_2 and x_1 are basic variables and s_1 and s_2 are non-basic variables.

$$x_B = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \quad x_N = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

Note that we keep the same order of variables as in the artificial problem in which we find them at the end of the problem.

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & 0 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 12 \\ 20 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \quad N = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad B^{-1}b = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$c_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad c_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_B^T B^{-1} b = 12$$

$$c_B^T B^{-1} N - c_N^T = [-1 \quad 0]$$