

4/1/2013

Artificial Variables (Cont'd)

We will continue solving a problem from last lecture.

Ex Find solution of the following problem:

$$P \left\{ \begin{array}{l} \min \quad x_1 + 2x_2 \\ \text{s.t.} \quad x_1 + 2x_2 \geq 12 \\ \quad \quad 2x_1 + 3x_2 \geq 20 \\ \quad \quad x_1, x_2 \geq 0 \end{array} \right.$$

We introduced surplus variables s_1, s_2 and artificial variables x_{a1}, x_{a2} and simplex method to solve problem (to find an initial feasible solution)

$$P_1 \left\{ \begin{array}{l} \min \quad x_{a1} + x_{a2} \\ \text{s.t.} \quad x_1 + 2x_2 - s_1 + x_{a1} = 12 \\ \quad \quad 2x_1 + 3x_2 - s_2 + x_{a2} = 20 \\ \quad \quad x_1, x_2, s_1, s_2, x_{a1}, x_{a2} \geq 0 \end{array} \right.$$

The optimal solution is achieved when $x_{a1} + x_{a2} = 0$, i.e. when both artificial variables are zero.

We obtained (last tableau of solving problem P_1):

	z	x_1	x_2	s_1	s_2	x_{a_1}	x_{a_2}	RHS
z	1	0	0	0	0	-1	-1	0
x_2	0	0	1	-2	1	2	-1	4
x_1	0	1	0	3	-2	-3	2	4

I_2 $B^{-1}N$ B^{-1} $B^{-1}b$

Now we can continue with solving the original problem P and we can re-use some of the information from the last tableau: B^{-1} , $B^{-1}N$, $B^{-1}b$.

We choose x_2, x_1 to be basic variables, s_1, s_2 : non-basic. We need to compute reduced cost and the value of the objective function for the original problem.

$$x_B = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \quad x_N = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & 0 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 12 \\ 20 \end{bmatrix}$$

\uparrow \uparrow
 x_1 x_2

$$B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \quad N = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad B^{-1}b = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$c_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad c_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note: we need to keep the same order of variables as in problem P_1 , so that we can re-use the last tableau of problem P_1 .
Note that variables x_2 and x_1 are switched.

Matrix B^{-1} is positioned under artificial variables. This is because we started (problem P_1) with $B = I$, identity matrix). As usual, the remainder of tableau holds $B^{-1}N$. We just need to compute reduced cost and value of the original objective function.
The last tableau can be read as:

	z	x_B	s	x_a	RHS
z	1	0	0	$-e$	0
x_2	0	I_2	$B^{-1}N$	B^{-1}	$B^{-1}b$
x_1					

or

	z	x_1	x_2	s_1	s_2	x_{a1}	x_{a2}	RHS
z	1	0	0	0	0	-1	-1	0
x_2	0	0	1	-2	1	2	-1	4
x_1	0	1	0	3	-2	-3	2	4
			I_2	$B^{-1}N$		B^{-1}		$B^{-1}b$

For the original problem we can compute

$$C_B^T B^{-1} b = 12$$

value of original
objective function

$$C_B^T B^{-1} N - C_N^T = [-1 \ 0]$$

reduced cost

and form a new tableau

	x_1	x_2	s_1	s_2	RHS
z	1	0	0	-1	0
x_2	0	0	1	-2	1
x_1	0	1	0	3	-2

Since we are solving a minimization problem and reduced cost is negative (-1) or zero, we conclude that we found an optimal solution $x_1 = 4, x_2 = 4, s_1 = s_2 = 0$.

Lemma The optimal objective function value in Problem P_1

$$P_1 \begin{cases} \min & e^T x_a \\ \text{s.t.} & Ax + I_m x_a = b \\ & x, x_a \geq 0 \end{cases}$$

is bounded below by 0. Furthermore, if the optimal solution to problem P_1 has $x_a = 0$, then the values of x form a feasible basic solution of problem P .

5

$$\square \quad e^T x_a = x_{a_1} \pi_1^0 + x_{a_2} \pi_2^0 + \dots + x_{a_m} \pi_m^0$$

Clearly, setting $x_a = 0$ will produce an objective function value of zero. Since $e = (1, 1, \dots, 1)^T > 0$, we cannot obtain a smaller objective function value. If at optimality we have $x_a = 0$, then we know that m variables have been chosen to be in the basis and the remaining variables (from x and x_a) are not in basis and hence they are zero. Therefore, we found a basic feasible solution. \square

4/3/2013

Thm Let x, x_a be an optimal feasible solution to problem P . Problem P is feasible if and only if $x_a = 0$.

□ \Leftarrow

We already proved in the previous lemma that if $x_a = 0$, then x is a feasible solution

\Rightarrow Let Problem P be feasible \Rightarrow there exists at least ^{one} basic feasible solution.

This is assured by a result that every polyhedral set has a nonzero finite # of extreme points. Now we can simply put $x_a = 0$ and x be the basic feasible solution to problem P . This is clearly an optimal solution of problem P , since it gives 0 value to the objective function $e^T x_a$ (that has 0 as a lower bound) □

The Two-Phase Simplex Algorithm

This algorithm summarizes what we discussed previously and gives an end-to-end algorithm that can be used to solve a LP problem.

(1) Given a problem in the form of a general maximization (or minimization) problem, convert it to the standard form:

$$P \begin{cases} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases}$$

with $b \geq 0$.

(2) Introduce auxiliary variables x_a and solve the Phase I problem:

$$P_1 \begin{cases} \min & e^T x_a \\ \text{s.t.} & Ax + I_m x_a = b \\ & x, x_a \geq 0 \end{cases}$$

$$\begin{cases} \dots + x_{a_1} = b_1 \\ \dots + x_{a_2} = b_2 \end{cases}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{a_1} \\ x_{a_2} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

(3) If $x_a = 0$, then an initial basic solution has been identified. This solution can be converted to a basic feasible solution of problem P as will be discussed. Otherwise, there is no solution to problem P.

(4) Use the basic feasible solution identified in step 3 to start the Simplex Method (compute the reduced cost given by c vector and the value of the objective function of

problem P).

(5) Solve the Phase II problem

$$P \begin{cases} \max c^T x \\ \text{s.t. } Ax \leq b \\ x \geq 0 \end{cases}$$

When we solve Phase I problem and find $x_a \neq 0$ at optimality, then there is no solution. If $x_a = 0$, then there are two possibilities:

- (1) The basis consists only of variables in vector x : no auxiliary variables x_a is in the basis.
- (2) There is some auxiliary variable $x_a = 0$ and this variable is in the basis. The solution is degenerate and degeneracy is expressed in an auxiliary variable.

Case (1): $x_a = 0$ and it is out of the basis. Then we have identified a basic feasible solution $x = [x_B \ x_N]^T$. Simply allow non-zero variables (in x) to be x_B and the remainder of the elements to be x_N . Then you can Phase II using

this initial basic feasible solution.

Case (2) : $x_a = 0$ but it is not out of the basis, i.e. there is at least one auxiliary variable still in the basis and this gives a degenerate solution of Problem P_1 . One approach is to keep artificial variable x_a in and solve Phase \bar{u} problem but never allow variable x_a to become positive. A better approach is to pivot and switch x_a with (some) non-basic variables x_N . This will remove an artificial variable x_a from the basis.

To remove the artificial variables, we can rewrite the tableau by rearranging row from 1 to m :

	x_B	x_{Ba}	x_N	x_{Na}	RHS
x_B	I_k	0	R_1	R_3	\bar{b}
x_{Ba}	0	I_{m-k}	R_2	R_4	0

4/5/2013

Two-Phase Simplex Method (Cont'd)

Case (2): $x_a = 0$ but x_a is in the basis.

Goal: to remove x_a from the basis by swapping x_a with one of the components of x_N

By swapping columns (this will change the order of variables), we can write the tableau in the form

	x_B	x_{Ba}	x_N	x_{Na}	RHS
(i) x_B	I_k	0	R_1	R_3	\bar{b}
x_{Ba}	0	I_{m-k}	R_2	R_4	0

$\underbrace{\hspace{10em}}_{B^{-1}N}$

$m-k$
artificial
basic
variables

Artificial variables x_a are split in basic artificial variables x_{Ba} and non-basic artificial variables x_{Na} .

We would like to switch variables x_{Ba} with x_N . We will attempt to pivot on elements of matrix R_2 .

We split matrix $B^{-1}N$ into

R_1	R_3
R_2	R_4

$$x_{Ba} = \begin{bmatrix} x_{Ba_1} \\ x_{Ba_2} \end{bmatrix} \quad x_N = \begin{bmatrix} x_{N_1} \\ x_{N_2} \end{bmatrix}$$

$$R_2 = \begin{bmatrix} (R_2)_{11} & (R_2)_{12} \\ (R_2)_{21} & (R_2)_{22} \end{bmatrix} : 2 \times 2 \text{ matrix}$$

	x_B	x_{Ba}	x_{N_1}	x_{N_2}	x_{Na}	RHS
x_B	I_k	0	R_1		R_3	\bar{b}
x_{Ba_1} x_{Ba_2}	0	I_{m-k}	$\begin{bmatrix} (R_2)_{11} & (R_2)_{12} \\ (R_2)_{21} & (R_2)_{22} \end{bmatrix}$		R_4	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We can pivot only on non-zero elements of R_2 . For example, if (1,1) elements of R_2 is non-zero, i.e. $(R_2)_{11} \neq 0$, then we can swap variables x_{N_1} and x_{Ba_1} . Then artificial variable x_{Ba_1} will be out of the basis. The value of x_{N_1} is zero since it is non-basis. Since RHS value that corresponds to x_{N_1} is zero from MRT, the value of x_{N_1} will not change (it will remain 0) and the value of the objective function will not change either.

MRT: $\min \frac{b_i}{a_{ji}}$ but b_i that corresponds to x_{Ba_1} is 0

This will produce a new tableau of similar form as in (1) but with $m-k-1$ artificial basic variables and $k+1$ artificial non-basic variables.

In executing this procedure, one of two things will occur:

- (1) Matrix R_2 will be transformed into I_{m-k}
- (2) a point will be reached when there is no variable from x_N that can enter the basis. This will happen when all elements of R_2 are zero.

In case (1), we removed all artificial variables from the basis in Phase I and we can proceed to Phase II with the current basic feasible solution.

In case (2) we have shown that

$$A \sim \begin{bmatrix} I_k & R_1 \\ 0 & 0 \end{bmatrix}$$

can be reduced

This shows that $m-k$ rows are not linearly independent of A

independent of first k rows and thus

matrix A did not have the full rank.
We can discard the last $m-k$ rows of A
and simply proceed with solution

$$x_B = \bar{b}, \quad x_N = 0.$$

This is a basic feasible solution to the
new matrix A in which we removed
the redundant rows.

Ex Once Phase I is complete, we have to
compute reduced cost and value of the
objective function from Phase II. This
can be computed during Phase I by
adding an extra "z" row.

We will consider again the problem

$$\begin{array}{l} \text{min} \quad x_1 + 2x_2 \\ \text{s.t.} \quad x_1 + 2x_2 \geq 12 \\ \quad \quad 2x_1 + 3x_2 \geq 20 \\ \quad \quad x_1, x_2 \geq 0 \end{array}$$

(See lectures from 3/29, 4/1). A new
tableau will have the form:

	z	x_1	x_2	s_1	s_2	x_{a_1}	x_{a_2}	RHS
z_{II}	1	-1	-2	0	0	0	0	0
z	1	3	5	-1	-1	0	0	32
x_{a_1}	0	1	2	-1	0	1	0	12
x_{a_2}	0	2	3	0	-1	0	1	20

Note: the first row (z_{II}) corresponds to the objective function

(2) $x_1 + 2x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot x_{a_1} + 0 \cdot x_{a_2}$

To complete row z_{II} , you need to compute

$C_B^T B^{-1} b$ value of objective function and $C_B^T B^{-1} N - C_N^T$ reduced cost

using objective function z_{II} from (2)
The entries in row (z) come from

$C_B^T B^{-1} b$ and $C_B^T B^{-1} N - C_N^T$
but

$$z = e^T x_a = x_{a_1} + x_{a_2}$$

i. cost vector c are different in Phase I and phase II problems.

Phase I: $c = (1 \ 1 \ 0 \ 0 \ 0 \ 0)^T$
 $\uparrow x_{a_1} \quad \uparrow x_{a_2}$

Phase II: $c = (1 \ 2 \ 0 \ 0 \ 0 \ 0)^T$

$\uparrow \quad \uparrow$
 $x_1 \quad x_2$

Part of tableau in is the same as in Phase I solution.

If we proceed exactly as when we solved Phase I part, we will get a series of tableau:

TABLEAU I

	z	x_1	x_2	s_1	s_2	x_{a_1}	x_{a_2}	RHS	MRT(x_i)
z_{II}	1	-1	-2	0	0	0	0	0	
z	1	3	5	-1	-1	0	0	32	
x_{a_1}	0	1	2	-1	0	1	0	12	12/1
x_{a_2}	0	2	3	0	-1	0	1	20	20/2=10

TABLEAU II

	z	x_1	x_2	s_1	s_2	x_{a_1}	x_{a_2}	RHS	MRT(x_i)
z_{II}	1	0	-1/2	0	-1/2	0	1/2	10	
z	1	0	1/2	-1	1/2	0	-3/2	2	
x_{a_1}	0	0	1/2	-1	1/2	1	-1/2	2	$\frac{2}{1/2} = 4$
$x_{a_2} \rightarrow x_1$	0	1	3/2	0	-1/2	0	1/2	10	$\frac{10}{3/2} = \frac{20}{3} \approx 6.6..$

TABLEAU III

	z	x_1	x_2	s_1	s_2	x_{a_1}	x_{a_2}	RHS
z_{II}	1	0	0	-1	0	1	0	12
z	1	0	0	0	0	-1	-1	0
$x_{a_1} \rightarrow x_2$	0	0	1	-2	1	2	-1	4
x_1	0	1	0	3	-2	-3	2	4

We arrive at the end of Phase I since all components of reduced cost are either 0 or negative. But now we are prepared to solve Phase II problem:

	z	x_1	x_2	s_1	s_2	RHS	MRT(s_2)
z_{II}	1	0	0	-1	0	12	
x_2	0	0	1	-2	1	4	$\frac{4}{1} = 4$
x_1	0	1	0	3	-2	4	

$B^{-1}N$ $-B^{-1}A_{.y}$

We are already at an optimal solution since components of reduced cost for z_{II} are 0 or negative. Note that since component of reduced cost that corresponds to s_2 is 0, we have alternative optimal solutions.

MRT on s_2 tells us that s_2 can change on $[0, 4]$ without changing the value of objective function.

$$x = B^{-1}b - B^{-1}N x_N$$

$$x_1 = 4, x_2 = 4, \quad B^{-1}b = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \quad b = \begin{bmatrix} 12 \\ 20 \end{bmatrix}$$

We only use s_2 in x_N : $x = B^{-1}b - B^{-1}A_{\cdot 4} \cdot s_2$

$$s_2 \in [0, 4]$$

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & 0 & -1 \end{bmatrix}$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & s_1 & s_2 \end{array}$$

$$B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

Note columns of B
are flipped because
we have

x_2, x_1 order

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot s_2, \quad s_2 \in [0, 4]$$

$$\text{or } \begin{array}{l} x_2 = 4 - s_2 \\ x_1 = 4 + 2s_2 \end{array}, \quad s_2 \in [0, 4]$$