

4/8/2013

Exam 2 covers Simplex Method up to
and including Two Phase Simplex Method
(up to & including lecture from 4/5/2013)

Exam is on Monday, April 15th in class.

Review session: Sunday, April 14th 10-11 am.

The Big-M Method

The Big-M method is similar to the two-phase simplex algorithm, except that it essentially attempts to execute Phase I and Phase II in a single execution of the simplex algorithm.

In the Big-M method, we modify problem P with artificial variables as we did in the two-phase algorithm (we modify constraints) but we also modify the objective function:

$$P_M \left\{ \begin{array}{l} \max c^T x - M e^T x_a \\ \text{s.t. } Ax + I_m x_a = b \\ x, x_a \geq 0 \end{array} \right.$$

Here, M is a large number, much larger than the largest coefficient in vector c. The value of M is usually chosen to be at least 100 times larger than the largest coefficient in the original objective function.

Note. The objective function is modified by the large term $Mx^T x_a$ that is a large penalty for any solution with $x_a \neq 0$. This makes the states where $x_a \neq 0$ very unattractive from the objective function point of view and the simplex method will "want" to eliminate artificial variables x_a from the basis.

Remark In case of a minimization problem, the objective function in the Big-M becomes

$$\min c^T x + Mx^T x_a$$

Lemma Suppose that problem P_M is unbounded. Then if problem P is feasible (i.e. feasible region is non-empty) then problem P is also unbounded.

This means that if problem P_M is unbounded, then we don't have any useful information about (no useful solution to) problem P .

Thm If problem P is feasible and has a finite solution. Then there is a $M>0$ so that the optimal solution to P_M has all artificial variables non-basic (i.e. zero) and thus the solution to problem P can be extracted from the solution to problem P_M .

Ex We will solve the same problem as before
 but we will use the Big-M method.
 The problem is

$$\begin{aligned} \min \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \geq 12 \\ & 2x_1 + 3x_2 \geq 20 \\ & x_1, x_2 \geq 0 \end{aligned}$$

In standard form:

$$\begin{aligned} \min \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 - s_1 = 12 \\ & 2x_1 + 3x_2 - s_2 = 20 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

To execute big-M method, we'll choose $M=300$ - which is larger more than 100 times of the largest coefficient of $z = x_1 + 2x_2$. The new problem becomes:

$$\begin{aligned} \min \quad & x_1 + 2x_2 + 300x_{a_1} + 300x_{a_2} \\ \text{s.t.} \quad & x_1 + 2x_2 - s_1 + x_{a_1} = 12 \\ & 2x_1 + 3x_2 - s_2 + x_{a_2} = 20 \\ & x_1, x_2, s_1, s_2, x_{a_1}, x_{a_2} \geq 0 \end{aligned}$$

Note Since we solve minimization problem, we add $M e^T x_a$ to the objective function.

Letting x_{a_1}, x_{a_2} be our initial basis, we get a series of tableaux.

TABLEAU I

	\downarrow	\downarrow	\downarrow	\downarrow				RHS	MRT(x_1)
Z		x_1	x_2	s_1	s_2	x_{a_1}	x_{a_2}		
Z	1	899	1498	-300	-300	0	0	96.00	
x_{a_1}	0	1	2	-1	0	1	0	12	$\frac{12}{1} = 12$
x_{a_2}	0	2	3	0	-1	0	1	20	$\frac{20}{2} = 10$

TABLEAU II

	\downarrow	\downarrow	\downarrow	\downarrow				RHS	MRT(x_2)
Z		x_1	x_2	s_1	s_2	x_{a_1}	x_{a_2}		
Z	1	0	$299/2$	-300	$299/2$	0	$-899/2$	610	
x_{a_1}	0	0	$1/2$	-1	$1/2$	1	$-1/2$	2	$\frac{2}{1/2} = 4$
$x_{a_2} \rightarrow x_1$	0	1	$3/2$	0	$-1/2$	0	$1/2$	10	$\frac{10}{3/2} = 6.6$

TABLEAU III

	\downarrow	\downarrow	\downarrow	\downarrow				RHS
Z		x_1	x_2	s_1	s_2	x_{a_1}	x_{a_2}	
Z	1	0	0	-1	0	-299	-300	12
$x_{a_1} \rightarrow x_2$	0	0	1	-2	1	2	-1	4
x_1	0	1	0	3	-2	-3	2	Y

Note This is essentially the same series of tableaux we had when executing the Two-Phase method, but we had to deal with the large M' coefficients in our arithmetic (in the reduced cost and objective function values).

4/15/2013

Exam #2 will be on simplex method.

- Algebraic form of simplex method
- Tableau form
- Finite solution
- Alternative solutions
- Unbounded problem (including direction)
- Convergence
- Degenerate solution
- Initialization of simplex method
 - artificial variables
 - Two Phase Method
 - big M method
 - Two "Row 0" method

HW #9

#1

	Z	X_1	X_2	S_1	S_2	RHS
1	0	0	a	b	c	d
X_1	0	1	0	$-\frac{2}{3}$	$-\frac{1}{3}$	1
X_2	0	0	1	$-\frac{1}{3}$	$-\frac{2}{3}$	1

$$X_B = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad X_N = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \quad S_1 = S_2 = 0$$

(a) Reduced cost $[a \ b]$

Minimization problem: to decrease objective function, we need $\frac{\partial z}{\partial x_j} < 0 \Rightarrow$ if x_j changes from 0 to some positive value $\Rightarrow z \downarrow$

$$\frac{\partial z}{\partial x_j} < 0 \Rightarrow \text{reduced cost } -\frac{\partial z}{\partial x_j} > 0$$

At the optimal solution, they, reduced cost is $\leq 0 \Rightarrow a \leq 0$ or $b \leq 0$.

(b) Alternative optimal solution: $a=0$ or $b=0$

Do MRT on the variable for which reduced cost is zero to determine all alternative solutions.

(see lecture 7/5/2013, for example, pp. 7-8)

HW #8

#1

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array}$$

Q1 Under which condition should a non-basic variable enter the basis.

Q2 Under which condition the problem P is unbounded?

$$\underline{Q1} \quad \frac{\partial z}{\partial x_j} = g_j - \underbrace{c_B^T B^{-1} A_{\cdot j}}_{z_j}$$

Reduced cost is $-\frac{\partial z}{\partial x_j} = z_j - g_j$

For min problem, a variable should enter
the basis if $\stackrel{\wedge}{z_j} < 0$ or reduced cost > 0 .

$$\underline{Q2} \quad x_B = B^{-1} b - B^{-1} N x_N, \quad x_N = 0$$

When a variable x_j enters the basis,
solution becomes

$$x_B = \underbrace{B^{-1} b}_{\bar{b} \geq 0} - \underbrace{B^{-1} A_{\cdot j} \cdot x_j}_{\bar{a}_j} \geq 0$$

If all components of \bar{a}_j are ≤ 0 , then
since $\bar{b} \geq 0$, $x_j \geq 0$, components of $x_B \geq 0$
for all possible values of x_j ; i.e. we can
increase value of x_j without making x_B
negative. In this, the value of objective
function \uparrow without any bound and
problem is unbounded.

Hence, a finite solution is possible only if $\bar{a}_j > 0$ (at least one component).

Lemma

(HW#8 #1) For a minimization problem, if $\bar{a}_{j_i} \leq 0$ $i=1,..,m$ and reduced cost $\bar{z}_j - \bar{c}_j > 0$ then problem P is unbounded.

(#3)

$$\min z(x_1, x_2) = 2x_1 - x_2$$

$$\text{s.t. } x_1 - x_2 + s_1 = 1$$

$$2x_1 + x_2 - s_2 = 6$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Reduce problem to standard form and implement simplex Method.

	\downarrow	\downarrow				
z	x_1	x_2	s_1	s_2	RHS	
1	-4	0	0	1	$\frac{2}{3}$	
s_1	0	3	0	1	$\frac{7}{3}$	
x_2	0	2	1	0	$\frac{18}{3}$	

We see that s_2 should enter the basis but $\bar{a}_j < 0$ (all components are negative). We have unbounded solution.

$$X_B = B^{-1} b - \underbrace{B^{-1} A_{0y}}_{\text{II}} S_2$$

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$X_B = \begin{bmatrix} 7 \\ 18/3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} S_2$$

extreme direction

$$d = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \leftarrow -\bar{q}_j$$

$$e_2 \leftarrow \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x_N = \begin{bmatrix} x_1 \\ s_2 \end{bmatrix}$$

4/19/2013

The Single Artificial Variable Technique

Consider the system of equations $AX = b$ for solving a problem P.

Typically we split matrix A in basic matrix B and non-basic matrix N:

$A = [B \mid N]$, and we require $x_B = B^{-1}b \geq 0$. If matrix A has a full rank, then such solution exists. It is not always possible to choose matrix B such that $\bar{b} = B^{-1}b \geq 0$. If matrix A has full rank, we can always find B such that B^{-1} exists but $B^{-1}b$ may not be ≥ 0 .

If $\bar{b} = B^{-1}b \geq 0$, then we have identified (by luck) an initial basic feasible solution and we can proceed with simplex Method as we did before.

We call
basis B
a crash
basis

Suppose $\bar{b} = B^{-1}b \neq 0$. Then we can form a new system:

$$I_m x_B + B^{-1}N x_N + \underbrace{\vec{y}_a}_{\text{new / added}} x_a = \bar{b} \quad (1)$$

Here

\vec{y}_a is a (column) vector

new / added

$$y_a = \begin{cases} -1, & b_i < 0 \\ 0, & \text{otherwise} \end{cases}$$

Note: eq⁴ (1) was obtained from $AX = b$ by splitting $A = [B|N]$ and multiplying both sides by B^{-1} . Then $\vec{y}_a x_a$ was added

$$AX = b \Rightarrow B^{-1} | BX_B + NX_N = b$$

$$I_m \vec{x}_B + B^{-1} N \vec{x}_N = B^{-1} b$$

Lemma Suppose we enter variable x_a into the basis by pivoting on the row of the simplex tableau with the most negative right hand side. That is, x_a is exchanged with variable x_{B_j} having most negative value. Then resulting solution is a basic feasible solution to the constraints

$$(2) \quad \begin{cases} I_m \vec{x}_B + B^{-1} N \vec{x}_N + \vec{y}_a x_a = \vec{b} \\ \vec{x}, x_a \geq 0 \end{cases}$$

↑
vector of coefficients
for artificial variable x_a

Note The single artificial variable technique can be thought of as multiple artificial variable techniques (we add one artificial

variable to each constraint) but with equal value. Hence this approach is more restrictive and less efficient than Two Phase Method or Big-M.

The resulting basic feasible solution can either be used as a starting solution for the Two Phase method with single artificial variable or the Big-M method.

For Two-Phase Method, we would need to solve Phase I problem.

$$(3) \begin{array}{ll} \min & \vec{x}_a \\ \text{s.t.} & \vec{A}\vec{x} + \vec{B}_0\vec{y}_a \vec{x}_a = \vec{b} \end{array}$$

where \vec{B}_0 is our initial crash basis.
Eq (3) is obtained by multiplying constraint (2) by \vec{B}_0 .

Ex We would like to study the following constrained set of equations.

$$x_1 + 2x_2 - s_1 = 12$$

$$2x_1 + 3x_2 - s_2 = 20$$

We can choose the crash basis:

$$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & 0 & -1 \end{bmatrix}$$

$$x_B = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \quad x_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

This choice corresponds to letting s_1, s_2 to be basic variables ($\neq 0$) and $x_1, x_2 = 0$ (non-basic).

Then since $x_1 = x_2 = 0$, our solution

$$s_1 = -12, \stackrel{<0}{\text{---}} \quad s_2 = -20 \stackrel{<0}{\text{---}}$$

We will rewrite the system as

$$-x_1 - 2x_2 + s_1 = -12$$

$$-2x_1 - 3x_2 + s_2 = -20$$

We can introduce an artificial variable x_a and vector $\vec{y}_a = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ (since both

components of RHS are negative) to obtain a new system:

$$-x_1 - 2x_2 + s_1 - x_a = -12$$

$$-2x_1 - 3x_2 + s_2 - x_a = -20$$

We can build tableau for Phase I problem with the current basic feasible solution.

	\downarrow	\downarrow	\downarrow			
Z	x_1	x_2	s_1	s_2	x_a	RHS
Z	1	0	0	0	-1	0
s_1	0	-1	-2	1	0	-1
s_2	0	-2	-3	0	1	-20

We enter variable x_a and pivot out variable s_2 which has the most negative right hand side to obtain the initial basic feasible solution.

Z	x_1	x_2	s_1	s_2	x_a	RHS
Z	1	2	3	0	-1	0
s_1	0	1	1	1	-1	0
$s_2 \rightarrow x_a$	0	2	3	0	-1	1

Then $s_1 = 8^{>0}$, $x_a = 20^{>0}$: basic feasible solution. We can complete Phase I by regular approach until we drive x_a from the basis and reduce RHS to 0. At this point we have identified a basic feasible solution and once Phase I is complete, we can solve Phase II problem.

4/26/2013

Problems that Can't be Solved by Hand

So far we considered small size problems for which it is possible to identify solution or at least initial basic feasible solution by graphical approach. These problems hardly require Phase I method.

We will consider an example of multi-period inventory control problem.

This type of problems can generate many variables and constraints, even in small problems.

Ex McLearey's Shamrock Emporium produces and sells shamrocks for three days: the day before St. Patrick's Day, St. Patrick's Day and the day after St. Patrick's Day. This year, McLearey had 10 shamrocks left over from last year's sale. This year he expects to sell 100 shamrocks the day before St. Patrick's Day, 200 shamrocks the day of St. Patrick's Day and 50 shamrocks the day after St. Patrick's Day.

It costs McLearey \$2 to produce each shamrock and \$0.01 to store a shamrock over night. Additionally, McLearey can put shamrocks into long term storage for \$0.05 per shamrock.

McLearey can produce at most 150 shamrocks per day. Shamrocks must be produced within two days of being sold (or put into long term storage) otherwise, they wilt. Assuming that McLearey must meet his daily demand and will not start producing shamrocks early, he wants to know how many shamrocks he should make and store on each day to minimize his cost.

Let time be a parameter: time runs over three days.

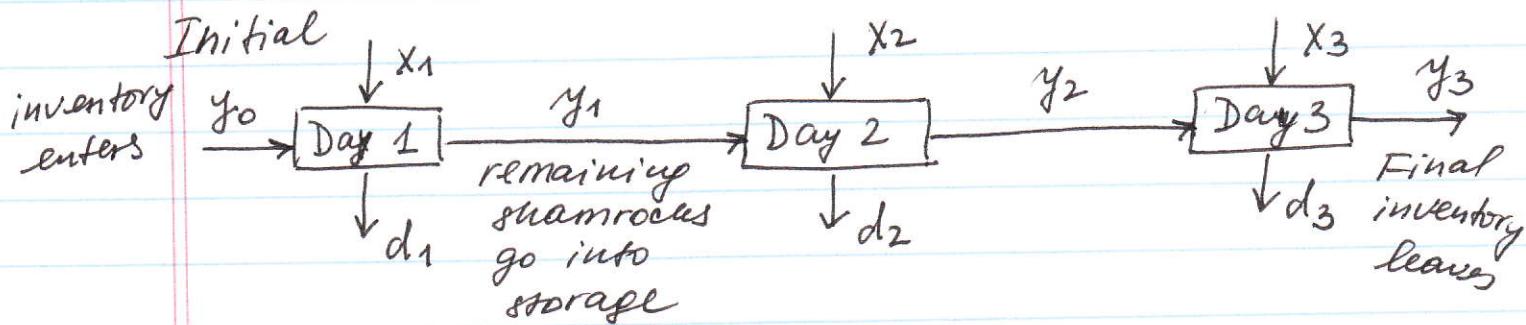
Let x_t be # of shamrocks company makes on day t ($t=1, 2, 3$).

Let y_t be # of shamrocks that McLearey stores on day t . We also have $y_0 = 10$: # of shamrocks left over from last year.

The objective function (cost in cents):

$$z = 200x_1 + 200x_2 + 200x_3 + y_1 + y_2 + 5y_3$$

There are other constraints linking production, storage and demand.



Multiperiod inventory models operate on a principle of conservation of flow.

Manufactured goods and previous period inventories flow into the box representing each day. Demand and next period inventories flow out of the box.

This inflow and outflow must be equal to account for all shamrocks.

$$y_0 + x_1 = d_1 + y_1$$

$$y_1 + x_2 = d_2 + y_2 \quad \text{or} \quad y_{t-1} + x_t = d_t + y_t$$

$$y_2 + x_3 = d_3 + y_3$$

for any t

Other constraints: $x_t \geq 0$

$$y_t \geq 0$$

Note: A negative inventory is a backorder.

By saying $y_t \geq 0$ means that McLeaney satisfies the demand every day.

Also: $x_t \leq 150, t=1, 2, 3$

The complete problem is

$$\begin{aligned} & \min && 200x_1 + 200x_2 + 200x_3 + y_1 + y_2 + 5y_3 \\ & \text{s.t.} && y_{t-1} + x_t = d_t + y_t, \quad t=1, 2, 3 \\ & && x_t \leq 150, \quad t=1, 2, 3 \\ & && x_t \geq 0, \quad y_t \geq 0 \end{aligned}$$

This "simple" problem has 6 variables $x_t, y_t, t=1, 2, 3$ and 6 constraints, and 6 non-negativity constraints.

Matlab can be used to solve such type of problems.

Another software: GLPK

GNU linear programming Kit

Solution: $x_1 = 140, x_2 = 150, x_3 = 50$
 $y_1 = 50, y_2 = 0, y_3 = 0$

Note inventory is zero in Year 3 (because it is expensive to store shamrocks and there is no information about future sales. This models "end of the world situation" and end of the world occurs immediately after day 3. This is typical for multi-period problems.

Linear Optimization

Math 326 Spring 2013

Prof. Lyudmyla Barannyk

Exam 2

April 17, 2013

Please SHOW ALL YOUR WORK and circle the final answers. If you need additional space, continue your work on the back of the page or extra sheet at the end of the exam. You can use the textbook and/or your notes.

Problem	Possible points	Score
1	20	
2	30	
3	50	
Total	100	

1. [20 pts] True/False Questions. Justify your answer to get a full credit.

- (a) [10 pts] True or False: Suppose x_j is a non-basic variable in some iteration of the simplex algorithm for some linear programming problem. Suppose $z_j - c_j < 0$ and there is a non-zero minimum ratio when performing the minimum ratio test. Then entering x_j will increase the objective function value.

$z_j - c_j < 0$: reduced cost

$$z_j - c_j = -\frac{\partial z}{\partial x_j} < 0 \Rightarrow \frac{\partial z}{\partial x_j} > 0$$

$\Rightarrow z \uparrow$ as $x_j \uparrow$

non-zero min. ratio implies that x_j will increase from $x_j=0$ to $x_{j\max} \neq 0 \Rightarrow z \uparrow$

True

- (b) [10 pts] True or False: When a Phase I problem has achieved optimality all the artificial variables are always out of the basis.

1) When Phase I problem achieves optimality, an artificial variable x_a may not be even zero. In this case, the original problem (Phase II) has no solution.

2) Phase I achieves optimality with $x_a = 0$ and $x_a \neq 0$ may be out of the basis as non-basic variable or $x_a = 0$ but it is not out of basis. In the latter case, one of approaches is to pivot on x_a ($x_a=0$, so value of the objective function will not increase) to swap x_a with one of physical variables.

False

2. [30 pts] Suppose I provide you with the following Tableau:

z	z	x_1	x_2	s_1	s_2	RHS
1	0	0	-1	-1	2	
x_1 ?	0	1	0	$-\frac{2}{3}$	$-\frac{1}{3}$	1
x_2 ?	0	0	1	$-\frac{1}{3}$	$-\frac{2}{3}$	1

- (a) Which variable names go in the places of the question marks?
- (b) If I tell you this is an **optimal** tableau, then is this a maximization problem or a minimization problem?
- (c) Now suppose I tell you this tableau is for a maximization problem. Is the problem bounded?

(a) $x_1 \ x_2$

(b) If tableau is optimal and reduced cost has negative components \Rightarrow this is a minimization problem. reduced cost $< 0 \Rightarrow \frac{\partial z}{\partial x_j} > 0 \Rightarrow$ optimal means that we can't decrease more.

(c) If tableau is for a maximization problem \Rightarrow should do MRT, but $B^{-1}A_j$ have only negative components

$$x_B = \underbrace{B^{-1}b}_{\geq 0} - \underbrace{B^{-1}A_j}_{< 0} x_j \geq 0 \text{ always}$$

and x_j can be increased as much as we wish and still $x_B \geq 0$.

\Rightarrow Problem is unbounded

3. [50 pts] Consider the problem:

$$\begin{aligned} \max \quad & 2x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 8 \\ & x_1 \leq 4 \\ & x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

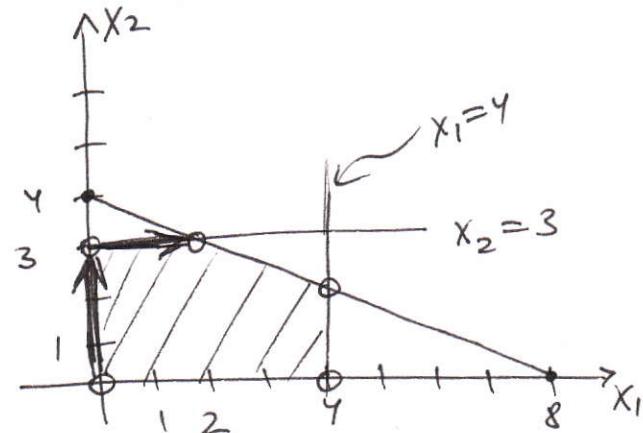
- (a) Sketch the feasible region. Circle the extreme points of the polyhedral set. (You do not need to sketch the level curves of the objective. You do not need to label the extreme points, just circle them.)
- (b) Can you determine from your sketch whether the problem will have a bounded solution or not?
- (c) Put the problem in standard form.
- (d) Set up a simplex tableau and execute **one iteration** of the simplex algorithm.
- (e) Illustrate, the step you just took using the simplex algorithm on your sketch of the feasible region.
- (f) [10 bonus points] if you find the optimal solution.

$$Z = 2x_1 + 5x_2$$

(b) feasible region is bounded
 \rightarrow problem is bounded

(c) $\max 2x_1 + 5x_2$
 s.t. $x_1 + 2x_2 + s_1 = 8$
 $x_1 + s_2 = 4$
 $x_2 + s_3 = 3$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$



$$X_B = B^{-1}b - B^{-1}N X_N$$

$$X_B = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix}$$

$$X_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B^{-1}N = N = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad C_N = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$b = \begin{bmatrix} 8 \\ 4 \\ 3 \end{bmatrix} \quad B^{-1} b = b = \begin{bmatrix} 8 \\ 4 \\ 3 \end{bmatrix}$$

$$C^T B^{-1} b = 0$$

value of the

objective function

$$\underbrace{C^T B^{-1} N - C_N^T}_{\text{"}} = -C_N^T = [-2 \quad -5]$$

	x_1	x_2	s_1	s_2	s_3	RHS	MRT (x_2)
Z	-2	-5	0	0	0	0	
S_1	1	2	1	0	0	8	$8/2 = 4$
S_2	0	0	0	1	0	4	$4/1 = 4$
S_3	0	0	0	0	1	3	$3/1 = 3 \leftarrow$

Let's choose x_2 to enter the basis.

	x_1	x_2	s_1	s_2	s_3	RHS	MRT (x_1)
Z	-2	0	0	0	5	15	
S_1	1	0	1	0	-2	2	$2/1 = 2 \leftarrow$
S_2	0	1	0	1	0	4	$4/1 = 4$
X_2	0	1	0	0	1	3	

	x_1	x_2	s_1	s_2	s_3	RHS	
Z	0	0	2	0	1	19	
X_1	0	1	0	1	0	-2	2
S_2	0	0	-1	1	2	2	
X_2	0	1	0	0	1	3	

We are at optimality since reduced cost $[2 \ 1]$ is positive \Rightarrow

$$z_{opt} = 19$$

$$x_1 = 2, x_2 = 3$$

2. [30 pts] Suppose I provide you with the following Tableau:

	z	x_1	x_2	s_1	s_2	RHS
	1	0	0	-1	-1	2
x_1 ?	0	1	0	$-\frac{2}{3}$	$-\frac{1}{3}$	1
x_2 ?	0	0	1	$-\frac{1}{3}$	$-\frac{2}{3}$	1

- (a) Which variable names go in the places of the question marks?
 (b) If I tell you this is an **optimal** tableau, then is this a maximization problem or a minimization problem?
 (c) Now suppose I tell you this tableau is for a maximization problem. Is the problem bounded?

(b) reduced cost is $[-1 \ -1]$

all components are negative \rightarrow min. problem

$$\text{red. cost} < 0 \Rightarrow \frac{\partial z}{\partial x_j} \geq 0$$

at optimality, this means that we can't make z smaller \rightarrow we have min. problem

(c) for a maximization problem, reduced cost being negative, implies that z can be increased. Nett we do MRT on s_1 or s_2

$$x_B = \underbrace{B^{-1}b}_{\substack{b \\ \geq 0}} - \underbrace{B^{-1}A_j x_j}_{\substack{\text{assumed} \\ \bar{a}_j}} \geq 0 \quad \checkmark \text{ need}$$

if all components of \bar{a}_j are negative one can increase x_j as much as he/she wants without causing x_B to become < 0 . \Rightarrow problem is unbounded

For both s_1 and s_2

$$\bar{a}_{s_1} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} \text{ and } \bar{a}_{s_2} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \text{ have all negative components}$$

Problem hence is unbounded

3. [50 pts] Consider the problem:

$$\begin{aligned} \max \quad & 2x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 8 \\ & x_1 \leq 4 \\ & x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- (a) Sketch the feasible region. Circle the extreme points of the polyhedral set. (You do not need to sketch the level curves of the objective. You do not need to label the extreme points, just circle them.)
- (b) Can you determine from your sketch whether the problem will have a bounded solution or not?
- (c) Put the problem in standard form.
- (d) Set up a simplex tableau and execute **one iteration** of the simplex algorithm.
- (e) Illustrate, the step you just took using the simplex algorithm on your sketch of the feasible region.
- (f) [10 bonus points] if you find the optimal solution.

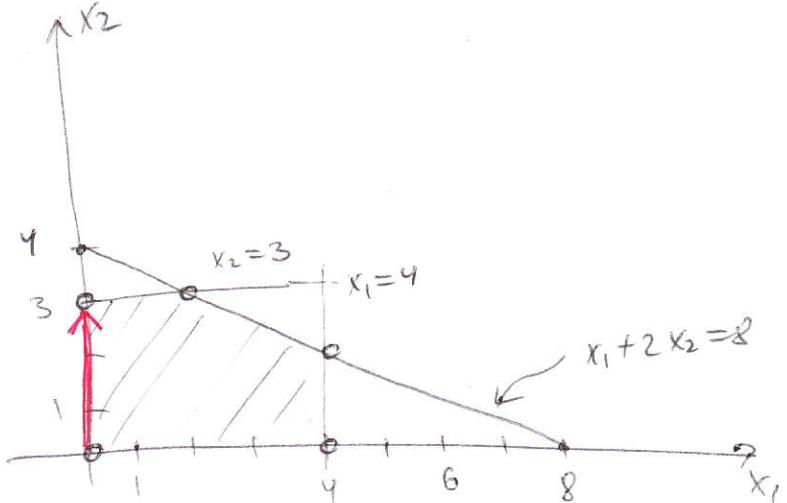
(a) $Z = 2x_1 + 5x_2$

(b) Feasible region is

Bounded \Rightarrow problem is
bounded.

(c)

$$\left\{ \begin{array}{l} \max Z = 2x_1 + 5x_2 \\ \text{s.t. } x_1 + 2x_2 + s_1 = 8 \\ \quad x_1 + s_2 = 4 \\ \quad x_2 + s_3 = 3 \\ \quad x_1, x_2, s_1, s_2, s_3 \geq 0 \end{array} \right.$$



(d)

Z	x_1	x_2	s_1	s_2	s_3	RHS	MRT (x_2)
1	-2	-5	0	0	0	0	
s_1	0	1	2	1	0	8	$\frac{8}{2} = 4$
s_2	0	1	0	0	1	4	
s_3	0	0	1	0	0	3	$\frac{3}{1} = 3 \leftarrow$

$$X_B = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \quad X_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad C_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad C_N = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 8 \\ 4 \\ 3 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B^{-1}N = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C^T B^{-1} b = 0 \quad C^T B^{-1} N - C_N^T = [-2 \quad -5]$$

$$B^{-1} b = \begin{bmatrix} 8 \\ 4 \\ 3 \end{bmatrix}$$

$\downarrow z$	x_1	x_2	s_1	s_2	s_3	RHS	MRT(x_1)
1	-2	0	0	0	5	15	
s_1	0	1	0	1	0	-2	2 ←
s_2	0	1	0	0	1	4	4
x_2	0	0	1	0	0	3	

$\downarrow z$	x_1	x_2	s_1	s_2	s_3	RHS
1	0	0	2	0	1	19
x_1	0	1	0	1	0	2
s_2	0	0	-1	1	0	2
x_2	0	0	1	0	1	3

Optimal tableau since reduced cost coeffs. > 0

$$\Rightarrow x_1 = 2, x_2 = 3 \quad \text{and } z_{\text{opt}} = 19$$

4/29/2013

Degeneracy and Convergence

Recall

Thm Consider Problem P (our linear programming problem). Let $B \in \mathbb{R}^{m \times m}$ be a basis matrix corresponding to some set of basic variables x_B . Let $\bar{b} = B^{-1} b$. If $\bar{b}_j = 0$ for some $j=1, \dots, m$, then $x_B = \bar{b}$ and $x_N = 0$ is a degenerate extreme point of the feasible region of problem P.

$$A \in \mathbb{R}^{m \times n} : \text{rank}(A) = m$$

We have seen that having a degenerate extreme point can cause extra # of step in executing simplex algorithm, but it may happen that for certain entering and leaving rules, simplex method would cycle without stopping. This is called stalling. In each step when a degenerate point is reached, simplex method would just give another representation of the same degenerate point without moving to another extreme pt.

Ex Demonstrate cycling [due to Seale]

$$\min -\frac{3}{4}x_4 + 20x_5 - \frac{1}{2}x_6 + 6x_7$$

$$\text{s.t. } x_1 + \frac{1}{4}x_4 - 8x_5 - x_6 + 9x_7 = 0$$

$$x_2 + \frac{1}{2}x_4 - 12x_5 - \frac{1}{2}x_6 + 3x_7 = 0$$

$$x_3 + x_6 = 1$$

$$x_i \geq 0, \quad i=1, \dots, 7$$

$$A = \left[\begin{array}{ccccccc} 1 & 0 & 0 & \frac{1}{4} & -8 & -1 & 9 \\ 0 & 1 & 0 & \frac{1}{2} & -12 & -\frac{1}{2} & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \right]$$

$\underbrace{\hspace{10em}}$

$$B = I$$

$$I$$

\Downarrow A contains identity matrix, so we could choose x_1, x_2, x_3 as initial basic variables.

$$B^{-1} = I$$

$$\text{Reduced cost: } C_B^T B^{-1} N - C_N^T = \left[\frac{3}{4} \quad -20 \quad \frac{1}{2} \quad -6 \right]$$

The initial tableau is

Tableau I

	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
Row 0	Z	1	0	0	$\frac{3}{4}$	-20	$\frac{1}{2}$	-6	0
Row 1	x_1	0	1	0	$\frac{1}{4}$	-8	-1	9	0
Row 2	x_2	0	0	1	$\frac{1}{2}$	-12	$-\frac{1}{2}$	3	0
Row 3	x_3	0	0	0	1	0	0	1	$\frac{1}{2}$

Entering rule: we always choose the non-basic variable with the most positive reduced cost.

Hence we choose x_4 to enter.

Leaving rule: the leaving variable will be the one for which the first row is in a tie.

In this case MRT gives the same value 0 for rows that correspond to variables x_1 and x_2 . We choose row 1 and we pivot about element $\frac{1}{4}$.

Tableau II

	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
Z	1	-3	0	0	0	4	$\frac{7}{2}$	-33	0
x_4	0	4	0	0	1	-32	-4	36	0
$\rightarrow x_2$	0	-2	1	0	0	$\frac{1}{4}$	$\frac{3}{2}$	-15	0
x_3	0	0	0	1	0	0	1	0	1

Tableau III

									RHS	MRT(x_6)
$\rightarrow z$	z	x_1	x_2	x_3	x_4	$x_5 -$	x_6	x_7		
	1	-1	-1	0	0	0	2	-18	0	
$\rightarrow x_4$	0	-12	8	0	1	0	(8)	-84	0	$\frac{9}{8}$
x_5	0	-1/2	1/4	0	0	1	3/8	-15/4	0	$\frac{0}{3/8}$
x_3	0	0	0	1	0	0	1	0	1	$\frac{1}{1}$

Tableau IV

	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
$\rightarrow Z$	1	2	-3	0	$-1/4$	0	0	3	0
x_6	0	$-3/2$	1	0	$1/8$	0	1	$-2/2$	0
$\rightarrow x_5$	0	$1/16$	$-1/8$	0	$-3/64$	1	0	$3/16$	0
x_3	0	$3/2$	-1	1	$-1/8$	0	0	$21/2$	$\frac{1}{2}$

Tableau V

	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	MRT(x_1)
$\rightarrow x_6$	1	1	-1	0	$\frac{1}{2}$	-16	0	0	0	
x_7	0	$\boxed{2}$	-6	0	$-5/2$	56	1	0	0	$\frac{0}{2}$
x_3	0	$1/3$	$-2/3$	0	$-1/4$	$16/3$	0	1	0	$\frac{0}{1/3}$
	0	-2	6	1	$5/2$	-56	6	0	1	

Tableau VI

	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	MRT(x_2)
$\rightarrow Z$	1	0	2	0	$7/4$	-44	$-1/2$	0	0	
x_1	0	1	-3	0	$-5/4$	28	$1/2$	0	0	
$\rightarrow x_7$	0	0	$\boxed{1/3}$	0	$1/6$	-4	$-11/6$	1	0	
x_3	0	0	0	1	0	0	1	0	1	

Tableau VII

	Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS
Z	1	0	0	0	$\frac{3}{4}$	-20	$\frac{1}{2}$	-6	0
X_1	0	1	0	0	$\frac{1}{4}$	-8	-1	9	0
X_2	0	0	1	0	$\frac{1}{2}$	-12	$-\frac{1}{2}$	3	0
X_3	0	0	0	1	0	0	1	0	1

But this tableau is the same as Tableau I. Hence, simplex method would cycle forever without finishing its execution.

5/1/2013

The lexicographic Minimum Ratio Leaving Variable Rule

There are many methods that can be used to prevent cycling. One of them is lexicographic minimum ratio leaving variable rule.

Def Let $\vec{x} = [x_1, \dots, x_n]^T$, $\vec{y} = [y_1, \dots, y_n]^T$ be vectors in \mathbb{R}^n . We say that x is lexicographically greater than y if there is $m < n$ such that $x_i = y_i$ $i=1, \dots, m$ but $x_{m+1} > y_{m+1}$.

Note: if $x_i = y_i$ $i=1, \dots, n$, then $x = y$.

Notation: $x \succ y$: x is lexicographically greater than y

$x \succeq y$: x is lexicographically greater or equal than y

Def A vector x is lexicographically positive if $x \succ 0$, where 0 is zero vector.

Claim If vectors x and y are lexicographically positive, then so is $x+y$ and $c x$, where $c > 0$: constant.

lexicographic Minimum Ratio Test.

Assume we have some fixed entering variable rule and variable x_j enters the basis. Next we do minimum ratio test, which determines which variable leaves the basis. Consider

$$I_0 = \left\{ r : \frac{\bar{b}_r}{\bar{a}_{jr}} = \min \left[\frac{\bar{b}_i}{\bar{a}_{ji}} : i=1, \dots, m, \bar{a}_{ji} > 0 \right] \right\}$$

this is min ratio test. $\bar{b} = B^{-1}b$, $\bar{a}_j = B^{-1}A_{\cdot j}$.

In the absence of degeneracy, I_0 contains one element: index of the row/variable that leaves the basis.

If we have a degenerate solution, I_0 may have more than one elements. This means that at least two rows tie. In such case we consider another set:

$$I_1 = \left\{ r : \frac{\bar{a}_{1r}}{\bar{a}_{jr}} = \min \left[\frac{\bar{a}_{1i}}{\bar{a}_{ji}} : i \in I_0 \right] \right\}$$

Note Here we are taking elements of column 1 of $B^{-1}A_0$, to obtain \bar{a}_1 . The elements of this column are then divided by elements of column \bar{a}_j component-wise.

We use only those elements of \bar{a}_r which variables were tied in MRT. If set I_r contains only one element, then we know that variable x_{B_r} leaves the basis. Otherwise we use column 2 etc. In general

$$I_r = \{ r : \frac{\bar{a}_{kr}}{\bar{a}_{jr}} = \min \left[\frac{\bar{a}_{ki}}{\bar{a}_{ji}} : i \in I_{r-1} \right] \text{ and } \bar{a}_{ji} > 0 \}$$

Claim For any nondegenerate basis matrix B , we will ultimately find set I_k such that I_k contains only one element.

Note In executing lexicographic minimum ratio test, we essentially comparing the tied rows in lexicographic manner.

If a set of rows ties in the minimum ratio test, then we execute a minimum ratio test on the first column of tied rows. If there is a tie, then we move on executing a minimum ratio test on the second column of rows that were tied in both previous test. This continues until the tie is broken and a single row is found that gives the index of leaving variable.

Ex The same example by Seale (as in last lecture). Entering rule: we choose the variable with the largest in magnitude reduced cost. Leaving variable rule: lexicographic min ratio test.

Tableau I

	\bar{z}	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	MRT(x_4)
Row 0	z	1	0	0	$\frac{3}{4}$	-20	$\frac{1}{2}$	-6	0	
Row 1	x_1	0	1	0	$\frac{1}{4}$	-8	-1	9	0	$\frac{0}{\frac{1}{4}} = 0$
Row 2	x_2	0	0	1	$\frac{1}{2}$	-12	$-\frac{1}{2}$	3	0	$\frac{0}{\frac{1}{2}} = 0$
Row 3	x_3	0	0	0	1	0	0	1	1	

$$I_0 = \{1, 2\}$$

Next we complete

$$\min \left\{ \frac{1}{\frac{1}{4}}, \frac{0}{\frac{1}{2}} \right\} = 0 \text{ at Row 2}$$

$\Rightarrow I_1 = \{2\} \Rightarrow$ variable x_2 leaves the basis

We pivot about element $\frac{1}{2}$ to get

Tableau II


Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	MRT(x_6)
1	0	$-3/2$	0	0	-2	$5/4$	$-21/2$	0	
x_1	0	1	$-1/2$	0	0	-2	$-3/4$	$15/2$	0
x_4	0	0	2	0	1	-24	-1	6	0
x_3	0	0	0	1	0	0	1	0	$\frac{1}{1}$

We see that x_6 has to enter the basis and x_3 leaves the basis, and we obtain

Tableau III

Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
1	0	$-3/2$	$-5/4$	0	-2	0	$-21/2$	$-5/4$
x_1	0	1	$-1/2$	$3/4$	0	-2	0	$15/2$
x_4	0	0	2	1	1	-24	0	6
x_6	0	0	1	0	0	1	0	1

We are at optimal solution. since all reduced cost components are negative.