Lecture 10

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HW #3

#1 $y = f(x)$ $\quad x = x_0$

$$y(x) = y(x_0) + y'(x_0)(x-x_0) + \frac{y''(x_0)}{2!}(x-x_0)^2 + \ldots$$

$$\ldots + \frac{y^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{y^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} =$$

$$= P_n(x) + R_n(x) \quad : \text{Taylor Thm}$$

$$P_n(x) = y(x_0) + y'(x_0)(x-x_0) + \ldots + \frac{y^{(n)}(x_0)}{n!}(x-x_0)^n :$$ Taylor polynomial of degree n

$$R_n(x) = \frac{y^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} :$$ remainder

ξ is between x_0 and x

$$\underbrace{y(x)}_{\text{exact}} = \underbrace{P_n(x)}_{\text{approximation}} + \underbrace{R_n(x)}_{\text{error}}$$

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$y(x) \approx P_n(x)$ but error is given by $R_n(x)$

Ex $y(x) = e^x \qquad x_0 = 0$

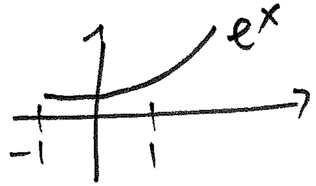
$y' = e^x, \quad y'' = y''' = \dots = e^x$

$$y(x) = \underbrace{\underset{=1}{e^0} + e^0 \cdot (x-0) + \frac{e^0}{2!} x^2 + \dots + \frac{e^0}{n!} x^n}_{P_n(x)} + \frac{e^{\xi}}{(n+1)!} x^{n+1}$$

If we want to approximate $y(x) = e^x$ using $P_n(x)$ with error $\varepsilon = 10^{-5}$, we require

$$\left| \frac{e^{\xi}}{(n+1)!} x^{n+1} \right| < \varepsilon = 10^{-5}$$

Let $x \in [-1, 1]$



$$\left| \frac{e^{\xi}}{(n+1)!} x^{n+1} \right| \le \max_{\xi \in [-1,1]} |e^{\xi}| \cdot \max_{x \in [-1,1]} |x^{n+1}| \cdot \frac{1}{(n+1)!} =$$

$$= \frac{e^1 \cdot 1^{n+1}}{(n+1)!} = \frac{e}{(n+1)!} < 10^{-5}$$

solve for n

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#3 need to use triangular inequality:

$$|x+y| \le |x| + |y|$$

or $|x-y| \le |x| + |y|$

need to show $\qquad x_{n+1} = g(x_n)$

$$|\alpha - x_{n+1}| \le \frac{k}{1-k} |x_{n+1} - x_n|$$

we showed in class:

$$|\alpha - x_{n+1}| = \underbrace{|g'(\xi)|}_{\le k} |\alpha - x_n| \le k|\alpha - x_n|$$

ie. $\boxed{|\alpha - x_{n+1}| \le k|\alpha - x_n|}$

$$|\alpha - x_{n+1}| \le k|\alpha - x_n| = k|(\alpha - x_{n+1}) + (x_{n+1} - x_n)| \underset{\text{inequality}}{\overset{\Delta}{\le}}$$

$$\le k(|\alpha - x_{n+1}| + |x_{n+1} - x_n|)$$

#2 $|\alpha - x_{n+1}| \le |g'(\xi)| \cdot |\alpha - x_n| \le \underbrace{k}_{\text{asympt. const}} \cdot |\alpha - x_n|$

$|g'(\xi)| \le \max |g'(\xi)| = k$

in practice you can use $k \approx |g'(\alpha)|$

Restoring quadratic convergence in Newton's or **secant methods**

Newton's method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ given x_0

- if root α is a simple root of $f(x)$, then
  $|\alpha - x_{n+1}| \le C|\alpha - x_n|^2$: quadratic convergence
- if root α is of multiplicity m, then
  $|\alpha - x_{n+1}| \le C|\alpha - x_n|$: linear convergence

Q How to restore quadratic convergence?

I method

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)}$$

where m is multiplicity of root α.

II method

Assume that α is a root of $f(x)$ of multiplicity m.
Then we can write

$$f(x) = (x-\alpha)^m h(x), \qquad h(\alpha) \neq 0$$

$$f'(x) = m(x-\alpha)^{m-1} h(x) + (x-\alpha)^m h'(x) =$$
$$= (x-\alpha)^{m-1} [m h(x) + (x-\alpha) h'(x)]$$

Define

$$F(x) = \frac{f(x)}{f'(x)} = \frac{(x-\alpha)^m h(x)}{(x-\alpha)^{m-1}\left[m\, h(x) + (x-\alpha)\, h'(x)\right]} =$$

$$= (x-\alpha)\, \frac{h(x)}{m\, h(x) + (x-\alpha)\, h'(x)}$$

$F(\alpha) = 0 \Rightarrow \alpha$ is a root of $F(x)$

One can show that $F'(\alpha) \neq 0 \Rightarrow \alpha$ is a simple root of $F(x)$. Therefore, we can apply Newton's method to function F and this will give quadratic convergence to α.

$F(x) = \dfrac{f(x)}{f'(x)}$: we do need to know multiplicity m to use this approach

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Summary: given $f(x)=0$ α is a root of $f(x)$

Consider

$$F(x) = \frac{f(x)}{f'(x)}$$

apply Newton's method to $F(x)$ to find α.

Secant method (restored convergence)

$$x_{n+1} = x_n - \frac{f(x_n)}{\dfrac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}} \Rightarrow x_{n+1} = x_n - m\,\frac{f(x_n)}{\dfrac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}}$$

where m is multiplicity of α.

Back to LU factorization.

Given $Ax = b$

$$L \underbrace{Ux}_{y} = b$$

1. $A = LU$

2. Solve $Ly = b$ for y

3. Solve $Ux = y$ for x

Ex

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

We found previously $m_{21} = -\frac{1}{2}$

$m_{31} = 0 \qquad m_{32} = -\frac{2}{3}$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{pmatrix}$$

We also reduced A to the upper triangular matrix

$$U = \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

Therefore, we can write

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}}_{U}$$

Solve $Ax = b$, $\quad b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

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$$\Rightarrow \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

To check: multiple $A \cdot x \stackrel{?}{=} b$

<u>Storage</u> (triangularization)

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \rightarrow \begin{pmatrix} a_{11}^{(1)} & \cdots & a_{1n}^{(1)} \\ m_{21} & \ddots & \\ \vdots & & a_{nn}^{(n)} \\ m_{n1} & \cdots \; m_{n,n-1} & \end{pmatrix}$$

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Solve $Ly = b$

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{pmatrix} \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{y} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{array}{l} y_1 = 1 \\ y_2 = \frac{1}{2} \\ y_3 = \frac{4}{3} \end{array}$$

$1 \cdot y_1 = 1 \Rightarrow y_1 = 1$

$-\frac{1}{2} \cdot \underbrace{y_1}_{=1} + 1 \cdot y_2 = 0 \Rightarrow y_2 = \frac{1}{2}$

$0 \cdot y_1 + (-\frac{2}{3}) \underbrace{y_2}_{=\frac{1}{2}} + 1 \cdot y_3 = 1 \Rightarrow$ solve for y_3

$$\Rightarrow y = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{4}{3} \end{pmatrix}$$

Solve $Ux = y$

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{4}{3} \end{pmatrix}$$

$x_3 = 1$

$x_2 = 1$

$x_1 = 1$

$\frac{4}{3} \cdot x_3 = \frac{4}{3} \Rightarrow x_3 = 1$

$\frac{3}{2} \cdot x_2 - \underbrace{x_3}_{=1} = \frac{1}{2}$