Lecture 13

<u>Pf</u>

$$x^T A_h x = \sum_{i,j=1}^{n} a_{ij} x_i x_j = \frac{1}{h^2}\Big\{ \sum_{i=1}^{n} \underbrace{(2 + c_i h^2)}_{\geq 2} x_i^2 - \sum_{i=1}^{n-1} x_i x_{i+1} -$$

$$- \sum_{i=2}^{n} x_i x_{i-1} \Big\} \geq \frac{1}{h^2}\Big\{ \sum_{i=1}^{n} 2 x_i^2 - 2\sum_{i=1}^{n-1} x_i x_{i+1} \Big\} =$$

$$= \frac{1}{h^2}\Big\{ x_1^2 + x_n^2 + \sum_{i=1}^{n-1} x_i^2 + \sum_{i=1}^{n-1} x_{i+1}^2 - 2\sum_{i=1}^{n-1} x_i x_{i+1} \Big\} =$$

$$= \frac{1}{h^2}\Big\{ \underbrace{x_1^2}_{\geq 0} + \underbrace{x_n^2}_{\geq 0} + \sum_{i=1}^{n-1} \underbrace{(x_i - x_{i+1})^2}_{\geq 0} \Big\} \geq 0 \Rightarrow x^T A_h x \geq 0 \text{ for any } x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

If $x^T A_h x = 0 \Rightarrow$

$$x_1^2 = 0$$

$$x_n^2 = 0$$

$$(x_i - x_{i+1})^2 = 0 \qquad i = 1, .., n-1$$

$$\Rightarrow x_1 = x_2 = \dots = x_n = 0 \Rightarrow x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\therefore$ $x^T A_h x = 0$ iff $x = 0 \Rightarrow A_h$ is positive definite ∎

Convergence

If $c(x) \geq 0$, then $\max\limits_{1 \leq i \leq n} |u_i - \varphi_i| \leq \underbrace{\frac{1}{96} \left|\max \varphi^{(4)}(x)\right|}_{C} h^2$

Error Analysis

$Ax = b$, $\quad A$: invertible

x : exact solution $\quad x = A^{-1} b$

$\tilde{x}$: approximate solution

$e = x - \tilde{x}$: error

$r = b - A\tilde{x}$

Claim : $Ae = r$

Pf

$$Ae = A(x - \tilde{x}) = Ax - A\tilde{x} = b - A\tilde{x} = r$$

Note

$$e = 0 \iff r = 0$$

However, small residual r <u>does NOT</u> always imply that error e is also small.

<u>**Def**</u> a <u>vector norm</u> $\|x\|$ has the following properties:

1. $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$
2. $\|\alpha x\| = |\alpha| \cdot \|x\|$, where α is a scalar
3. $\|x+y\| \leq \|x\| + \|y\|$: triangle inequality

<u>**Ex**</u> Let $x = (x_1, \ldots, x_n)^T$

$\|x\|_\infty = \max\{|x_i|, i = 1, \ldots, n\}$ $\quad$ l_∞ norm or ∞ norm

$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$ $\quad$ l_2 norm or 2 norm

$\|x\|_1 = \sum_{i=1}^{n} |x_i|$ $\quad$ l_1 norm or 1-norm

Ex $x = (1,1,1)^T$ $\qquad \|x\|_\infty = 1, \quad \|x\|_2 = \sqrt{3}, \quad \|x\|_1 = 3$

Check that $\|x\|_\infty$ is indeed a norm. $\qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Pf

1. Clearly $\|x\|_\infty \geq 0$

$\|x\|_\infty = 0 \iff \max_i |x_i| = 0 \iff |x_i| = 0 \iff x_i = 0 \iff x = 0$

2. $\|\alpha x\|_\infty \overset{def}{=} \max\{\underbrace{|\alpha x_i|}_{\text{real \#s}}, i=1,\dots,n\} = |\alpha| \cdot \max\{|x_i|, i=1,\dots,n\} =$

$= |\alpha| \cdot \|x\|_\infty$

3. $\|x+y\|_\infty = \max\{|x_i + y_i|, i=1,\dots,n\} \leq \max\{|x_i| + |y_i|, i=1,\dots,n\} \leq$

$\leq \max\{|x_i|, i=1,\dots,n\} + \max\{|y_i|, i=1,\dots,n\} = \|x\|_\infty + \|y\|_\infty$

Q $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\qquad \|x\| \overset{def}{=} |x_1 - 2x_2|$. Can this $\|x\|$ be a norm?

$x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow |x_1 - 2x_2| = |2 - 2\cdot 1| = 0$ but $x \neq 0$

$\Rightarrow \|x\| = |x_1 - 2x_2|$ does not define a norm!

Ex

$$\left(\begin{array}{cc|c} 1.01 & 0.99 & 2 \\ 0.99 & 1.01 & 2 \end{array}\right) \qquad x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : \text{ exact solution}$$

$$b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\tilde{x}_1 = \begin{pmatrix} 1.01 \\ 1.01 \end{pmatrix} \qquad e_1 = x - \tilde{x}_1 = \begin{pmatrix} -0.01 \\ -0.01 \end{pmatrix} \qquad r_1 = b - A\tilde{x}_1 = \begin{pmatrix} -0.02 \\ -0.02 \end{pmatrix}$$

$$\tilde{x}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad e_2 = x - \tilde{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad r_2 = b - A\tilde{x}_2 = \begin{pmatrix} -0.02 \\ 0.02 \end{pmatrix}$$

Thus, $\|e_1\|_\infty = 0.01 \qquad \|r_1\|_\infty = 0.02 \qquad \|x\|_\infty = 1, \quad \|b\|_\infty = 2$

$\|e_2\|_\infty = 1 \qquad \|r_2\|_\infty = 0.02$

$$\underset{\frac{0.01}{1}}{\frac{\|e_1\|_\infty}{\|x\|_\infty}} = \underset{\frac{0.02}{2}}{\frac{\|r_1\|_\infty}{\|b\|_\infty}} \qquad \underset{\frac{1}{1}}{\frac{\|e_2\|_\infty}{\|x\|_\infty}} = 100 \cdot \underset{\frac{0.02}{2}}{\frac{\|r_2\|_\infty}{\|b\|_\infty}}$$

Q What is the relation between relative error $\frac{\|e\|}{\|x\|}$ and relative residual $\frac{\|r\|}{\|b\|}$?

Def A *matrix norm* $\|A\|$ has the following properties:

1. $\|A\| \geq 0$ and $\|A\| = 0$ iff $A = 0$
2. $\|\alpha A\| = |\alpha| \cdot \|A\|$, where α is a scalar
3. $\|A + B\| \leq \|A\| + \|B\|$
4. $\|A \cdot B\| \leq \|A\| \cdot \|B\|$

Given a vector norm $\|x\|$, the *subordinate* (or *induced*) matrix norm is defined by

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

The subordinate matrix norm has an additional property:

5. $\|Ax\| \leq \|A\| \cdot \|x\|$ for all vectors x

Ex

$$A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} \qquad \|A\|_\infty = \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \underset{\text{show later}}{=} 6$$