Lecture 14

Ex $A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}$ $\qquad \|A\|_\infty \overset{\text{def}}{=} \max\limits_{x \neq 0} \dfrac{\|Ax\|_\infty}{\|x\|_\infty} = 6$

$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad Ax_1 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \dfrac{\|Ax_1\|_\infty}{\|x_1\|_\infty} = \dfrac{4}{1} = 4$

$x_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad Ax_2 = \begin{pmatrix} 5 \\ 5 \\ 1 \end{pmatrix} \Rightarrow \dfrac{\|Ax_2\|_\infty}{\|x_2\|_\infty} = \dfrac{5}{1} = 5$

$x_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad Ax_3 = \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix} \Rightarrow \dfrac{\|Ax_3\|_\infty}{\|x_3\|_\infty} = \dfrac{6}{1} = 6$

Claim $\|A\|_\infty = \max\limits_{i} \sum\limits_{j=1}^{n} |a_{ij}|$ ("max row sum")

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Pf

By def, $\|A\|_\infty \overset{def}{=} \max\limits_{x\neq 0} \dfrac{\|Ax\|_\infty}{\|x\|_\infty}$

Step 1

$$|(Ax)_i| = \left|\sum_{j=1}^{n} a_{ij}\cdot x_j\right| \le \sum_{j=1}^{n} |a_{ij}|\cdot|x_j| \le \|x\|_\infty \cdot \sum_{j=1}^{n} |a_{ij}|$$

i^{th} component of Ax

$$\Rightarrow |(Ax)_i| \le \|x\|_\infty \cdot \max_i \sum_{j=1}^{n} |a_{ij}|, \quad \forall i$$

$$\|Ax\|_\infty \le \|x\|_\infty \cdot \max_i \sum_{j=1}^{n} |a_{ij}|$$

$$\frac{\|Ax\|_\infty}{\|x\|_\infty} \le \max_i \sum_{j=1}^{n} |a_{ij}| \qquad \text{for } \forall x \neq 0$$

$$\|A\| \le \max_i \sum_{j=1}^{n} |a_{ij}|$$

Aside

$|2| < |-5|$

$\not\Downarrow$

$2 < -5$

Step 2

Suppose $\max\limits_{i} \sum_{j=1}^{n} |a_{ij}| = \sum_{j=1}^{n} |a_{pj}|$

Define $y = (y_1, \dots, y_n)^T$ with $y_j = \begin{cases} 1 & \text{if } a_{pj} \geq 0 \\ -1 & \text{if } a_{pj} < 0 \end{cases}$

$$\|A\|_\infty \overset{\text{def}}{=} \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \geq \frac{\|Ay\|_\infty}{\|y\|_\infty} = \|Ay\|_\infty \geq |(Ay)_p| = \left|\sum_{j=1}^{n} a_{pj} y_j\right| =$$

$$= \sum_{j=1}^{n} |a_{pj}| = \max_{i} \sum_{j=1}^{n} |a_{ij}|$$

$$\Rightarrow \|A\|_\infty \geq \max_{i} \sum_{j=1}^{n} |a_{ij}|$$

Combining two steps, we get $\|A\|_\infty = \max\limits_{i} \sum_{j=1}^{n} |a_{ij}|$ ∎

Note

1. $\|X\|_2 = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2} = \left(X^T \cdot X\right)^{1/2}$

$\|A\|_2 = \max_{X \neq 0} \frac{\|AX\|_2}{\|X\|_2} = \max_i \{ \sqrt{\lambda_i}$, where λ_i is an eigenvalue of $A^T A \}$

Recall λ is an e'value of matrix A with associated e'vector $X \neq 0$ if $AX = \lambda X$

e'values can be computed by solving characteristic eq^n

$\det(A - \lambda I) = 0$

2. $\|X\|_1 = \sum_{j=1}^{n} |x_j|$

$\|A\|_1 = \max_{X \neq 0} \frac{\|AX\|_1}{\|X\|_1} = \max_j \sum_{i=1}^{n} |a_{ij}|$ ("max column sum")

Thm

Let A be invertible, $Ax = b$

x: exact solution; $\tilde{x}$: approximation

$e = x - \tilde{x}$: error; $r = b - A\tilde{x}$: residual

Then

$$\frac{\|e\|}{\|x\|} \le \kappa(A) \cdot \frac{\|r\|}{\|b\|}$$

where $\kappa(A) = \|A\| \cdot \|A^{-1}\|$: <u>condition number</u> of matrix A relative to norm $\|\cdot\|$, i.e. relative error $\frac{\|e\|}{\|x\|}$ is bounded by $\kappa(A)$ times relative residual.

Proof

$$\|b\| = \|Ax\| \le \|A\| \cdot \|x\| \Rightarrow \frac{1}{\|x\|} \le \frac{\|A\|}{\|b\|}$$

$$Ae = r \Rightarrow e = A^{-1} r \Rightarrow \|e\| = \|A^{-1} r\| \le \|A^{-1}\| \cdot \|r\| \Rightarrow \|e\| \le \|A^{-1}\| \cdot \|r\|$$

$$\Rightarrow \frac{\|e\|}{\|x\|} \le \underbrace{\|A\| \cdot \|A^{-1}\|}_{\kappa(A)} \cdot \frac{\|r\|}{\|b\|} \Rightarrow \frac{\|e\|}{\|x\|} \le \kappa(A) \cdot \frac{\|r\|}{\|b\|} \quad \blacksquare$$

Recall

$$A = \begin{pmatrix} 1.01 & 0.99 \\ 0.99 & 1.01 \end{pmatrix} \qquad \|A\|_\infty = 2$$

$$A^{-1} = \begin{pmatrix} 25.25 & -24.75 \\ -24.75 & 25.25 \end{pmatrix} \qquad \|A^{-1}\|_\infty = 50$$

$$\kappa_\infty(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty = 2 \cdot 50 = 100$$

Note

$$I = A \cdot A^{-1}$$

$$\underbrace{\|I\|}_{1} = \|A \cdot A^{-1}\| \leq \|A\| \cdot \|A^{-1}\| = \kappa(A) \Rightarrow \boxed{\kappa(A) \geq 1}$$

Note

1. $Ax = b$, $A\tilde{x} = \tilde{b}$ $\Rightarrow$ $$\frac{\|x - \tilde{x}\|}{\|x\|} \leq \kappa(A) \cdot \frac{\|b - \tilde{b}\|}{\|b\|}$$

2. $Ax = b$, $\tilde{A}\tilde{x} = b$ $\Rightarrow$ $\dfrac{\|x - \tilde{x}\|}{\|\tilde{x}\|} \le \kappa(A) \cdot \dfrac{\|A - \tilde{A}\|}{\|A\|}$

Proof of 1 follows from Thm, proof of 2 – HW (?)

<u>Recall</u>

$A = \begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix}$ $\qquad \kappa_\infty(A) \sim 4 \qquad \varepsilon = 10^{-2} \Rightarrow \kappa_\infty(A) = 4.0404$

$A^{(1)} = E_1 A = \begin{pmatrix} \varepsilon & 1 \\ 0 & 1 - \frac{1}{\varepsilon} \end{pmatrix} \qquad \kappa_\infty(E_1 A) \sim \frac{1}{\varepsilon^2} \Rightarrow \tilde{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$E_1 P_1 A = \begin{pmatrix} 1 & 1 \\ 0 & 1 - \varepsilon \end{pmatrix} \qquad \kappa_\infty(E_1 P_1 A) \sim 4 \Rightarrow \tilde{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Gaussian elimination is <u>unstable</u> because $\kappa(A^{(k)}) \gg \kappa(A)$. Perturbations of $A^{(k)}$ due to roundoff errors are amplified by $\kappa(A^{(k)})$ instead of $\kappa(A)$.

Gaussian elimination with partial pivoting is <u>stable</u> because $K(A^{(k)}) \sim K(A)$.

Iterative Methods

$$Ax = b \iff x = Bx + C$$

$$x_{k+1} = Bx_k + C$$

B: iteration matrix

C: constant vector