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Lecture 16

Recall,

Given matrix, $Ap = \lambda p$, $p \neq 0$, λ is an eigenvalue of A with associated eigenvector p, $p \neq 0$.

$$Ap = \lambda p \iff (A - \lambda I)p = 0 \underset{p \neq 0}{\iff} \det(A - \lambda I) = 0$$

$f_A(\lambda) = \det(A - \lambda I)$: <u>characteristic polynomial</u> of A

Roots of $f_A(\lambda)$ are <u>eigenvalues of</u> A.

<u>Ex</u> If A is an upper triangular matrix, then the eigenvalues of A are its diagonal elements.

<u>Pf</u>

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ 0 & & & a_{nn} \end{pmatrix}$$

$$f_A(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ & a_{22}-\lambda & \cdots & a_{2n} \\ & & \ddots & \vdots \\ 0 & & & a_{nn}-\lambda \end{pmatrix} =$$

$$= (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$$

$f_A(\lambda) = 0 \Rightarrow \det(A-\lambda I) = 0 \Rightarrow \lambda = a_{ii}, \quad i = 1, \ldots, n.$ $\blacksquare$

<u>Explanation</u> of $\|e_{k+1}\|_\infty = \frac{1}{4}\|e_k\|_\infty$ for G-S.

We showed

$$e_{k+1} = Be_k \qquad B = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix}$$

$\lambda_1 = 0$ is an e'value w/ associated e'vector $\vec{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = \frac{1}{4}$ is — " — " — $\vec{p}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

To check: $A\vec{p}_1 \overset{?}{=} \lambda_1 \vec{p}_1$, $A\vec{p}_2 \overset{?}{=} \lambda_2 \vec{p}_2$.

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$$B\vec{p}_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \underbrace{0}_{\lambda_1} \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{p}_1} = \lambda_1 \vec{p}_1 \quad \checkmark$$

$$B\vec{p}_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} = \underbrace{\frac{1}{4}}_{\lambda_2} \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\vec{p}_2} = \lambda_2 \vec{p}_2 \quad \checkmark$$

Vectors $\vec{p}_1$ and $\vec{p}_2$ are linearly independent since they correspond to distinct e'values.

Let $e_0 = \underbrace{X}_{\text{exact}} - \underbrace{X_0}_{\text{initial guess}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{p}_2 - \vec{p}_1$

$$e_1 = Be_0 = B(\vec{p}_2 - \vec{p}_1) = B\vec{p}_2 - B\vec{p}_1 = \lambda_2 \vec{p}_2 - \cancel{\lambda_1}^{\,0} \vec{p}_1 = \lambda_2 \vec{p}_2$$

$$e_2 = Be_1 = B(\lambda_2 \vec{p}_2) = \lambda_2 B\vec{p}_2 = \lambda_2^2 \vec{p}_2$$

— — — — —

$$e_k = \lambda_2^k \vec{p}_2 = \left(\frac{1}{4}\right)^k \vec{p}_2$$

$$e_{k+1} = \lambda_2^{k+1} \vec{p_2} = \lambda_2 \cdot \underbrace{\lambda_2^k \vec{p_2}}_{e_k} \Rightarrow e_{k+1} = \lambda_2 e_k$$

$$\Rightarrow \|e_{k+1}\|_\infty = \tfrac{1}{4} \|e_k\|_\infty$$

<u>Def</u> $\rho(B) = \{\max|\lambda|$ where λ is an eigenvalue of matrix $B\}$: <u>spectral radius of B</u>

<u>Thm</u> If $\rho(B) < 1$, then $x_n \to x$ for all initial guesses x_0.

<u>Pf</u>
Assume that matrix B has e'values $\lambda_1, \lambda_2, .., \lambda_n$ w/ associated e'vectors $\vec{p_1}, \vec{p_2}, .., \vec{p_n}$, that form a basis in $\mathbb{C}^n$ (this is true, for example, when B is symmetric).

For any x_0, $e_0 = x - x_0 = \alpha_1 \vec{p_1} + \alpha_2 \vec{p_2} + \ldots + \alpha_n \vec{p_n}$

$$e_1 = Be_0 = B(\alpha_1 \vec{p_1} + \alpha_2 \vec{p_2} + \ldots + \alpha_n \vec{p_n}) = \alpha_1 \underbrace{B\vec{p_1}}_{\lambda_1 \vec{p_1}} + \alpha_2 \underbrace{B\vec{p_2}}_{\lambda_2 \vec{p_2}} + \ldots + \alpha_n \underbrace{B\vec{p_n}}_{\lambda_n \vec{p_n}} =$$

$= \alpha_1 \lambda_1 \vec{p_1} + \alpha_2 \lambda_2 \vec{p_2} + \dots + \alpha_n \lambda_n \vec{p_n}$

$e_2 = B e_1 = B(\alpha_1 \lambda_1 \vec{p_1} + \dots + \alpha_n \lambda_n \vec{p_n}) = \alpha_1 \lambda_1^2 \vec{p_1} + \dots + \alpha_n \lambda_n^2 \vec{p_n}$

$\cdots \quad - \quad - \quad - \quad -$

$e_k = \alpha_1 \lambda_1^k \vec{p_1} + \dots + \alpha_n \lambda_n^k \vec{p_n}$

$\|e_k\| = \|\alpha_1 \lambda_1^k p_1 + \dots + \alpha_n \lambda_n^k \vec{p_n}\| \leq |\alpha_1| \cdot |\lambda_1|^k \|p_1\| + \dots + |\alpha_n| \cdot |\lambda_n|^k \|p_n\|$

Since $\rho(B) < 1 \Rightarrow |\lambda_1| < 1, |\lambda_2| < 1, \dots, |\lambda_n| < 1$

$\Rightarrow |\lambda_1|^k \to 0, \; |\lambda_2|^k \to 0, \dots, |\lambda_n|^k \to 0$ as $k \to \infty$

$\Rightarrow \|e_k\| \to 0$ as $k \to \infty \Rightarrow x_k \to x$ as $k \to \infty$.

<u>Note</u> The proof also shows that

$\|e_{k+1}\| \lesssim \rho(B) \cdot \|e_k\|$ as $k \to \infty$

Recall $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ Then $B_J = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ $\rho(B_J) = \frac{1}{2}$

$B_{GS} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix}$ $\rho(B_{GS}) = \frac{1}{4}$

$$\begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = \lambda^2 - \left(\frac{1}{2}\right)^2 = 0 \Rightarrow \lambda = \pm\frac{1}{2} \Rightarrow \rho(B_J) = \frac{1}{2}$$

Summary

1. $\|e_{k+1}\| \leq \|B\| \cdot \|e_k\|$: always
2. $\|e_{k+1}\| \lessapprox \rho(B) \cdot \|e_k\|$: true if B is diagonalizable and $|\lambda_1| < 1, |\lambda_2| < 1, \ldots, |\lambda_n| < 1$

Q What is the relation between $\|B\|$ and $\rho(B)$?

1. $\rho(B)$ is not a norm.

Pf Take $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \rho(B) = 0$ but $B \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ ☒

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2. $\rho(B) \le \|B\|$ for any matrix norm

Pf Suppose that λ is an arbitrary e'value of B w/ associated e'vector $\vec{p}$, $\vec{p} \neq \vec{0}$: $B\vec{p} = \lambda\vec{p}$

Then

$$|\lambda| \cdot \|\vec{p}\,\vec{p}^{\,T}\| = \|\underbrace{\lambda\vec{p}}_{B\vec{p}}\,\vec{p}^{\,T}\| = \|B\vec{p}\,\vec{p}^{\,T}\| \le \|B\| \cdot \|\vec{p} \cdot \vec{p}^{\,T}\|$$

$\vec{p} \neq \vec{0} \Rightarrow \|\vec{p}\,\vec{p}^{\,T}\| \neq 0 \Rightarrow$ we can divide both sides by $\|\vec{p}\,\vec{p}^{\,T}\|$

$\Rightarrow |\lambda| \le \|B\|$ true for any λ

$$\Rightarrow \underbrace{\max|\lambda|}_{\rho(B)} \le \|B\| \quad \Rightarrow \quad \rho(B) \le \|B\| \quad \blacksquare$$

3. $\rho(B) = \lim_{k\to\infty} \|B^k\|^{1/k}$

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This is useful because $e_k = B^k e_0$.

Ex

$$B_J = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad \rho(B_J) = \frac{1}{2} \leq \|B_J\|_\infty = \frac{1}{2}$$

$$B_{GS} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix} \qquad \rho(B_{GS}) = \frac{1}{4} \leq \|B_{GS}\|_\infty = \frac{1}{2}$$