Lecture 18

Recall

A is strictly diagonally dominant (ssd) if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \qquad i = 1, \dots, n$$

Ex

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} : \text{sdd}$$

Thm

If A is strictly diagonally dominant, then

1. $\|B_J\|_\infty < 1$

2. $\|B_{GS}\|_\infty < 1$

Pf

1. $A = L + D + U$

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| = \underbrace{\sum_{j=1}^{i-1} |a_{ij}|}_{\equiv l_i} + \underbrace{\sum_{j=i+1}^{n} |a_{ij}|}_{\equiv u_i} = l_i + u_i$$

this is just notation

Then $\dfrac{l_i + u_i}{|a_{ii}|} < 1$ for $i = 1, \dots, n$

Recall,

$$B_J = -D^{-1}(L+U), \qquad D^{-1} = \text{diag}\left\{\frac{1}{a_{11}}, \frac{1}{a_{22}}, \dots, \frac{1}{a_{nn}}\right\}$$

$$\Rightarrow \|B_J\|_\infty = \max_i \left\{\frac{l_i + u_i}{|a_{ii}|}\right\} < 1 \quad \blacksquare$$

2. HW #7 Consider 2×2 case

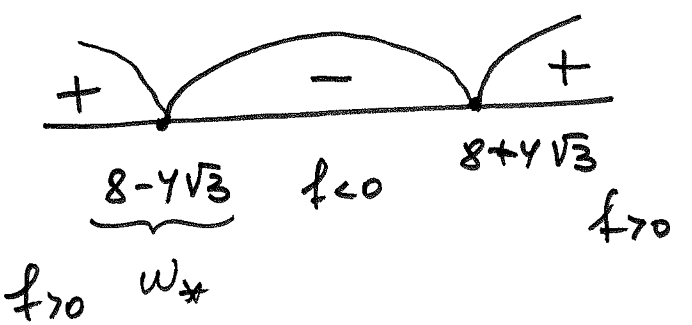
analysis of SOR iteration matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad B_\omega = \begin{pmatrix} 1-\omega & \frac{\omega}{2} \\ \frac{\omega(1-\omega)}{2} & \frac{\omega^2}{4} + 1 - \omega \end{pmatrix}$$

eigenvalues of B_ω (solutions of $\det(B_\omega - \lambda I) = 0$) are

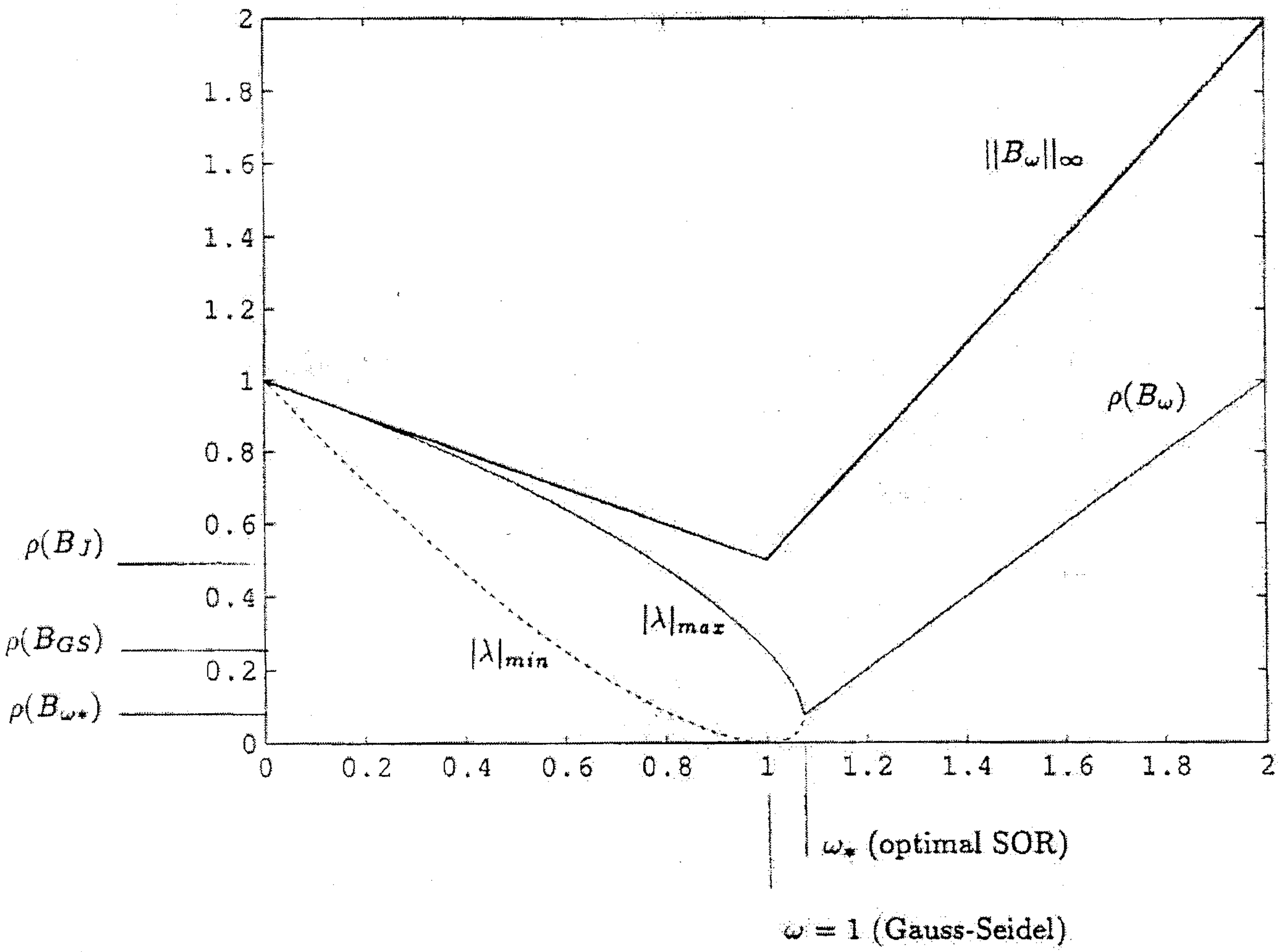
$$\lambda_{1,2} = 1 - \omega + \frac{1}{8}\omega^2 \pm \frac{1}{8}\omega\sqrt{16 - 16\omega + \omega^2}$$

$$f(\omega) = 16 - 16\omega + \omega^2$$



Analysis of the SOR iteration matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \implies B_\omega = \begin{pmatrix} 1-\omega & \omega/2 \\ \omega(1-\omega)/2 & \omega^2/4+1-\omega \end{pmatrix}$$



1. The optimal SOR parameter is $\omega_* = 4/(2+\sqrt{3}\,) \sim 1.0718$.

2. B_ω has 2 eigenvalues.

   For $\omega < \omega_*$, the eigenvalues are real and distinct, $|\lambda|_{min} < |\lambda|_{max}$.

   For $\omega = \omega_*$, there is a single real eigenvalue of multiplicity 2.

   For $\omega > \omega_*$, the eigenvalues are complex conjugate, $|\lambda|_{min} = |\lambda|_{max}$.

3. Note that $\rho(B_\omega) \le ||B_\omega||_\infty$. This is consistent with a theorem proven in class.

4. The SOR scheme converges for $0 < \omega < 2$ since $\rho(B_\omega) < 1$ for those parameter values.

5. The optimal SOR method converges more rapidly than either Jacobi or Gauss-Seidel:

   $\rho(B_J) = 0.5$

   $\rho(B_{GS}) = 0.25$

   $\rho(B_{\omega*}) = \omega_* - 1 \sim 0.0718$

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Nonlinear Systems (2×2)

$f(x,y) = 0$

$g(x,y) = 0$

$$f(x_{n+1}, \cancelto{0}{y_{n+1})} = f(x_n, y_n) + f_x(x_n, y_n)(x_{n+1} - x_n) + f_y(x_n, y_n)(y_{n+1} - y_n) \cancelto{0}{+ \ldots}$$

expand in Taylor series about (x_n, y_n)

$$g(x_{n+1}, \cancelto{0}{y_{n+1})} = g(x_n, y_n) + g_x(x_n, y_n)(x_{n+1} - x_n) + g_y(x_n, y_n)(y_{n+1} - y_n) \cancelto{0}{+ \ldots}$$

$$f_x(x_n, y_n)(x_{n+1} - x_n) + f_y(x_n, y_n)(y_{n+1} - y_n) = -f(x_n, y_n)$$

$$g_x(x_n, y_n)(x_{n+1} - x_n) + g_y(x_n, y_n)(y_{n+1} - y_n) = -g(x_n, y_n)$$

$$\underbrace{\left.\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}\right|_{(x_n, y_n)}}_{J\text{: Jacobian matrix}} \cdot \underbrace{\begin{pmatrix} x_{n+1} - x_n \\ y_{n+1} - y_n \end{pmatrix}}_{\equiv p} = - \underbrace{\begin{pmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{pmatrix}}_{\equiv v} \quad \text{given} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \quad (1)$$

$$\Rightarrow \quad J \cdot p = v$$

<u>Note</u> If we have only 1 eq^{n} $f(x)=0$, this method reduces to Newton's method.

Given $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, eq^{n} (1) is a Newton's method for a system of 2 equations.

<u>Note</u> We <u>do not</u> invert matrix to find $\begin{pmatrix} x_{n+1} - x_n \\ y_{n+1} - y_n \end{pmatrix}$ but solve the linear system $Jp = v$ using direct or iterative methods.

where $p = \begin{pmatrix} x_{n+1} - x_n \\ y_{n+1} - y_n \end{pmatrix}$ $\qquad p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$

Compute vector p at every iteration. Then update solution:

$$\Rightarrow \underbrace{\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}_n}_{\text{computed by solving } Jp = v} = \begin{pmatrix} x_{n+1} - x_n \\ y_{n+1} - y_n \end{pmatrix} \Rightarrow \begin{aligned} x_{n+1} &= x_n + (p_1)_n \\ y_{n+1} &= y_n + (p_2)_n \end{aligned}$$

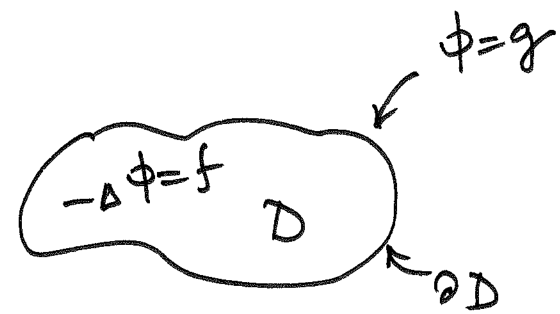
Note 1. ~~If~~ there is only one equation, this method reduces to Newton's method for scalar eq^n.

2. The method is sensitive to an initial guess.

Two-dimensional problems (BVP)

$$-\Delta \phi = f \quad \text{for} \quad (x,y) \in D \subset \mathbb{R}^2 :$$ **Poisson equation**

$$\Delta \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} : \text{ Laplace operator}$$



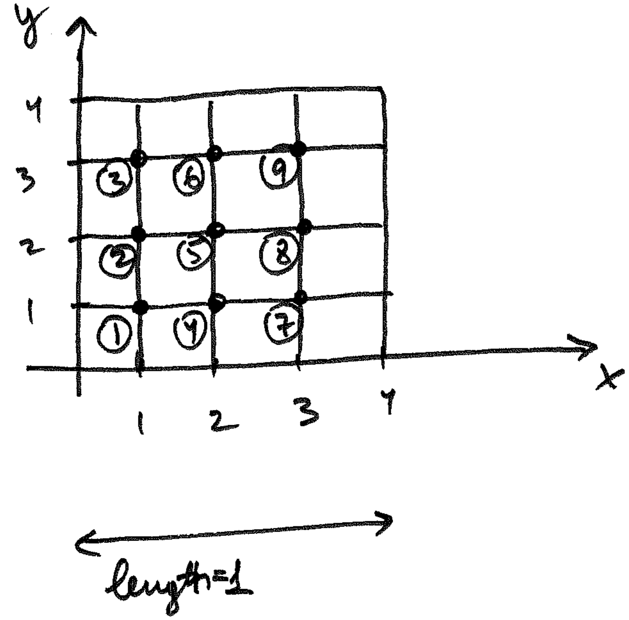
$\phi(x,y)$: unknown

$f(x,y)$: data - known

$\phi = g(x,y)$ for $(x,y) \in \partial\Omega$: Dirichlet BC

g known

Ex $D = [0,1] \times [0,1]$



Uniform mesh: (ih, jh),

$i, j = 0, 1, .., n+1, \quad h = \frac{1}{n+1}$

$\phi(ih, jh) \approx u_{ij}$

$f(ih, jh) = f_{ij}$

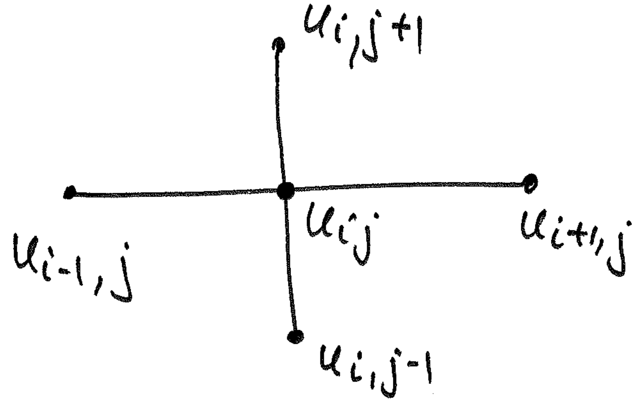
$$-\left(D_+^x D_-^x u_{ij} + D_+^y D_-^y u_{ij}\right) = f_{ij}$$

$$-\left(\frac{u_{i-1,j} - 2u_{ij} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{ij} + u_{i,j+1}}{h^2}\right) = f_{ij} :$$

<u>5 point discrete</u>
<u>Laplacian</u>

$$\frac{1}{h^2}\left(4u_{ij} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}\right) = f_{ij}$$

5 point stencil



Consider $\quad (i,j) = (1,1)$

$$\frac{1}{h^2}\left(4u_{11} - \underbrace{u_{01}}_{g_{01}} - u_{21} - \underbrace{u_{10}}_{g_{10}} - u_{12}\right) = f_{11}$$

$$\frac{1}{h^2}\left(4u_{11} - u_{21} - u_{12}\right) = f_{11} + \frac{1}{h^2}\left(g_{01} + g_{10}\right)$$