Lecture 19

Midterm Review

Review Taylor Thm

HW #6
(#3 d) we proved in class:

$Ax = b$

x: exact

$\tilde{x}$: approximation

$r = b - A\tilde{x}$

$e = x - \tilde{x}$: error

$$\underbrace{\frac{\|e\|}{\|x\|}}_{\text{relative error}} \le \underbrace{\kappa(A)}_{\text{condition \#}} \cdot \underbrace{\frac{\|r\|}{\|b\|}}_{\text{relative residual}}$$

$$f(x+h) = \underbrace{f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \dots + \frac{f^{(n)}(x)}{n!}h^n}_{P_n(x)} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1}}_{R_n(x)}$$

$P_n(x)$: Taylor polynomial of degree n

$R_n(x)$: remainder, ξ is between x and $x+h$

Ex $f(x) = \ln x$, $\quad x_0 = 2$, $\quad x \in [1, 3]$

$$f' = \frac{1}{x}, \quad f'' = -\frac{1}{x^2}, \quad f''' = \frac{2}{x^3}, \quad f^{IV} = \frac{-2\cdot 3}{x^4}, \quad \dots$$

$$f^{(n)}(x) = (-1)^{n+1} \frac{(n-1)!}{x^n}$$

$$\underset{\text{error}}{R_n(x)} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-2)^{n+1} = \underbrace{(-1)^{n+2}}_{(-1)^n \cdot \underbrace{(-1)^2}_{1} = (-1)^n} \frac{n!}{\xi^{n+1}} \cdot \frac{(x-2)^{n+1}}{(n+1)!}$$

~~Find~~ estimate n such that the error in approximating $f(x) = \ln x$ with n^{th} degree Taylor polynomial is $< 10^{-3}$.

$$|R_n(x)| = \frac{n!}{|\xi|^{n+1}} \cdot \frac{|x-2|^{n+1}}{(n+1)!} < \frac{n!}{(n+1)!} = \frac{n!}{n!(n+1)} = \frac{1}{n+1} < 10^{-3}$$

$1 \le x \le 3 \Rightarrow |x-2| \le 1$

$\xi \in [1, 3]$ $\quad \frac{1}{|\xi|} \le 1$

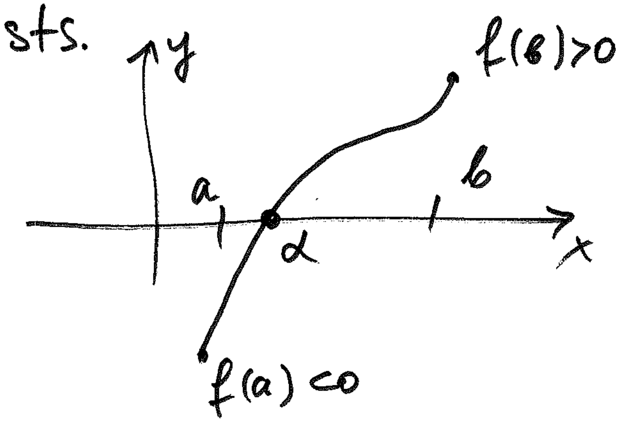
$1 \le |\xi| \le 3$

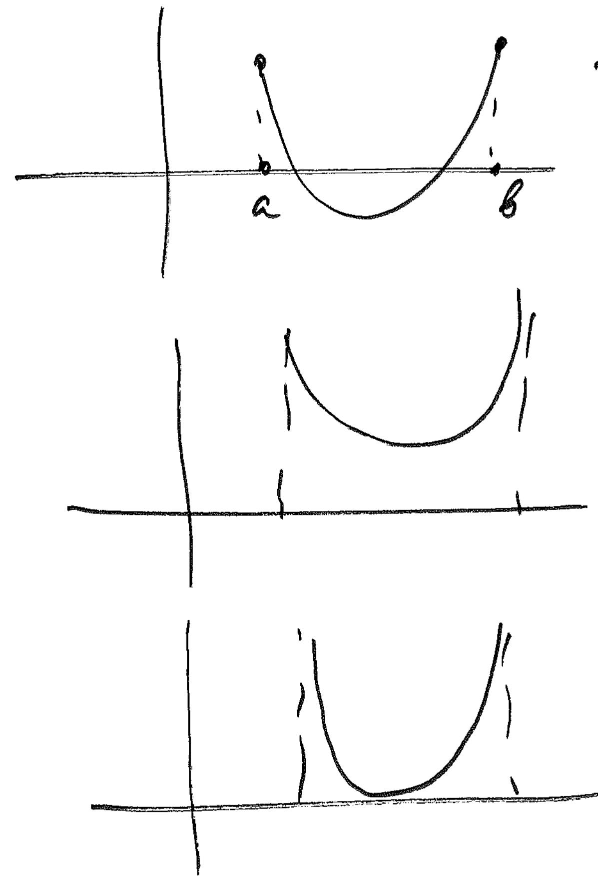
$n+1 > 1000$

$n > 999$

$\Rightarrow \boxed{n \geq 1000}$

Root finding methods: review conditions which guarantee that a root exists.





2 roots

no roots

1 root

Bisection method: linear convergence but guaranteed to converge

Newton's method: quadratic convergence for a simple root
linear — " — for a multiple root
sensitive to initial guess x_0

<u>Secant method</u> : superlinear convergence, sensitive to x_0

<u>Fixed point iterations</u> : $x_{n+1} = g(x_n)$ $\quad |g'(x)| \le k < 1$ for $\forall\, x \in [a, b]$

1.) g maps $[a, b]$ onto $[a, b]$
2) g is CTS
3) $|g'(x)| \le k < 1$

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iterative method: $x_{n+1} = g(x_n)$

$$E_{n+1} = \underset{g(\alpha)}{\underset{\parallel}{\alpha}} - \underbrace{x_{n+1}}_{g(x_n)} = g(\alpha) - g(x_n) =$$

$$g(x_n) = g(\alpha) + g'(\alpha)(x_n - \alpha) + \frac{g''(\alpha)}{2!}(x_n - \alpha)^2 + \ldots$$

$$= \cancel{g(\alpha)} - \left(\cancel{g(\alpha)} + g'(\alpha)(x_n - \alpha) + \frac{g''(\alpha)}{2!}(x_n - \alpha)^2 + \ldots \right) =$$

$$= -g'(\alpha)(x_n - \alpha) - \frac{g''(\alpha)}{2!}(x_n - \alpha)^2 + \ldots$$

$$\underbrace{\alpha - x_{n+1}}_{E_{n+1}} = g'(\alpha)\,\underbrace{(\alpha - x_n)}_{E_n} - \frac{g''(\alpha)}{2!}\,\underbrace{(\alpha - x_n)^2}_{E_n^2} + \dots$$

if $g'(\alpha) \neq 0 \Rightarrow$ $\alpha - x_{n+1} = g'(\xi)\,(\alpha - x_n)$ by Taylor Thm

$|\alpha - x_{n+1}| \leq \underset{\text{asympt. const}}{K} \cdot |\alpha - x_n| \iff |E_{n+1}| \leq K \cdot |E_n|$

where $|g'(\xi)| \leq K$

if $g'(\alpha) = 0$, but $g''(\alpha) \neq 0$

$$\Rightarrow \alpha - x_{n+1} = -\frac{g''(\xi)}{2!}\,(\alpha - x_n)^2$$

$|\alpha - x_{n+1}| \leq C \cdot |\alpha - x_n|^2$ where $\underset{\text{asympt. const}}{C} = \max \left|\frac{g''(\xi)}{2!}\right|$

- Review properties of norms. Which function, for example, cannot or can be norms?
- Gaussian elimination, pivoting
- Finite difference approximation

$$f' - D_+ f = O(h) \qquad h \to \frac{h}{2} \quad \Rightarrow \quad \text{error} \to \frac{1}{2}\,\text{error}$$

$$O(h^4) \qquad h \to \frac{h}{2} \quad \Rightarrow \quad \text{error} \to \left(\frac{1}{2}\right)^4 \text{error}$$