Lecture 20

Lexicographic ordering

$$
\begin{array}{ccc|ccc|ccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
u_{11} & u_{12} & u_{13} & u_{21} & u_{22} & u_{23} & u_{31} & u_{32} & u_{33} \\
4 & -1 & & -1 & & & & & \\
-1 & 4 & -1 & & -1 & & & & \\
 & -1 & 4 & & & -1 & & & \\
\hline
-1 & & & 4 & -1 & & -1 & & \\
 & -1 & & -1 & 4 & -1 & & -1 & \\
 & & -1 & & -1 & 4 & & & -1 \\
\hline
 & & & -1 & & & 4 & -1 & \\
 & & & & -1 & & -1 & 4 & -1 \\
 & & & & & -1 & & -1 & 4
\end{array}
$$

$Au = f$

$$A = \frac{1}{h^2}\begin{pmatrix} T & -I & O \\ -I & T & -I \\ O & -I & T \end{pmatrix} :$$

block tridiagonal
symmetric
positive matrix

Note

$$\begin{pmatrix}
4 & -1 & \boxed{0} & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & \boxed{0} & -1 & 0 & 0 & 0 & 0 \\
\boxed{0} & -1 & 4 & \boxed{0} & \boxed{0} & -1 & 0 & 0 & 0 \\
-1 & \boxed{0} & \boxed{0} & 4 & -1 & \boxed{0} & -1 & 0 & 0 \\
0 & -1 & \boxed{0} & -1 & 4 & -1 & \boxed{0} & -1 & 0 \\
0 & 0 & -1 & \boxed{0} & -1 & 4 & \boxed{0} & \boxed{0} & -1 \\
0 & 0 & 0 & -1 & \boxed{0} & \boxed{0} & 4 & -1 & \boxed{0} \\
0 & 0 & 0 & 0 & -1 & \boxed{0} & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & \boxed{0} & -1 & 4
\end{pmatrix}$$

Def If $a_{ij} = 0$ for $|i-j| > p$, A is called a **band matrix** and $2p+1$ is **bandwidth**.

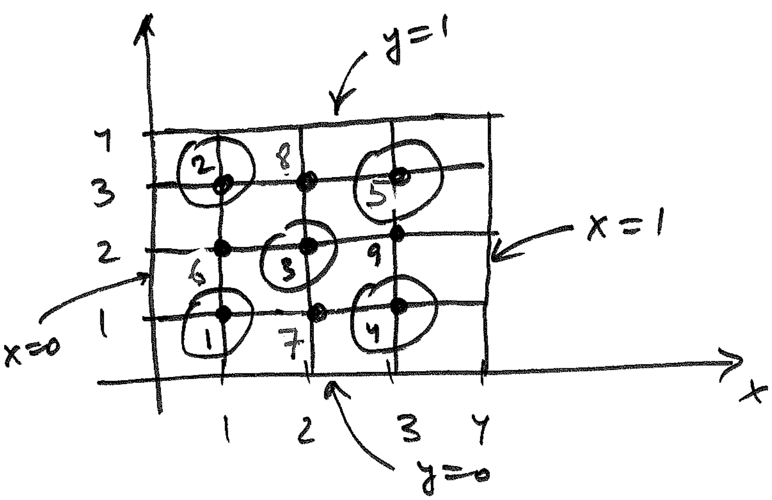
Ex $p = 3 \Rightarrow 2p+1 = 7$

Note as a result of LU factorization, fill-in occurs in the elements $\square$. LU factorization of a band matrix preserves the structure, but sparcity within the band will be lost. We have already seen this for a tridiagonal matrix.

Operation count

$$\# \text{mult} \sim np^2 \ll \frac{n^3}{3}$$

Red-black ordering



$$\begin{pmatrix} R & B & R \\ B & R & B \\ R & B & R \end{pmatrix}$$

$$
\begin{array}{ccccc|cccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
u_{11} & u_{13} & u_{22} & u_{31} & u_{33} & u_{12} & u_{21} & u_{23} & u_{32} \\
4 & & & & & -1 & -1 & & \\
 & 4 & & & & -1 & & -1 & \\
 & & 4 & & & -1 & -1 & -1 & -1 \\
 & & & 4 & & & -1 & & -1 \\
 & & & & 4 & & & -1 & -1 \\
\hline
-1 & -1 & -1 & & & 4 & & & \\
-1 & & -1 & -1 & & & 4 & & \\
 & -1 & -1 & & -1 & & & 4 & \\
 & & -1 & -1 & -1 & & & & 4
\end{array}
$$

H

D_R

D_B

K

With red-black ordering equations that result from application of 5-point discrete Laplacian can be written as

$$\begin{pmatrix} D_R & H \\ K & D_B \end{pmatrix} \begin{pmatrix} u_R \\ u_B \end{pmatrix} = \begin{pmatrix} b_R \\ b_B \end{pmatrix}$$

where D_R, D_B are scalar diagonal matrices. Then

$$\boxed{\begin{aligned} u_R &= -D_R^{-1} H u_B + D_R^{-1} b_R \\ u_B &= -D_B^{-1} K u_R + D_B^{-1} b_B \end{aligned}}$$

all red points can be computed/updated in parallel using black points. then all black points can be calculated using red points.

Polynomial Approximation (Chapter 5)

Def A polynomial $p(x)$ of degree n has the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n , \quad a_n \neq 0$$

The highest power, n in this case, is called the degree of the polynomial.

Ex $p(x) = 1 + x + x^2$ is a polynomial of degree 2

Note

1. Any polynomial is a continuous function.
2. But not all continuous functions are polynomials.

Ex $\sin x$

$f(x) = \dfrac{1}{1+x^2}$ is continuous but it is not a polynomial (f is a rational function)

Thm (Weierstrass)

Given a continuous function $f(x)$, $x \in [a, b]$. For any $\varepsilon > 0$, there exists a polynomial $p(x)$ such that

$$|f(x) - p(x)| \leq \varepsilon \quad \text{for all} \quad x \in [a, b]$$

$$\max_{a \leq x \leq b} |f(x) - p(x)| \leq \varepsilon$$

Application

$$\left| \int_a^b f(x)\,dx - \int_a^b p(x)\,dx \right| = \left| \int_a^b (f(x) - p(x))\,dx \right| \leq$$

$$\leq \int_a^b |f(x) - p(x)|\,dx \leq \int_a^b \varepsilon\,dx = \varepsilon(b-a).$$

Taylor Thm

Let $f(x)$ be defined on $[a,b]$ and suppose $f^{(n+1)}(x)$ is continuous for all $x \in [a,b]$. Then if $x, x_0 \in [a,b]$, then there exists $\xi = \xi(x)$ between x_0 and x such that

$$f(x) = p_n(x) + r(x)$$

where

$$p_n(x) = f(x_0) + f'(x_0)(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n :$$

n th degree Taylor polynomial at pt $x = x_0$

$$r(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} : \text{remainder or error}$$

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Ex $f(x) = \dfrac{1}{1+25x^2}$

$p_0 = 1$

$p_2 = 1 - 25x^2$

$p_4 = 1 - 25x^2 + 625x^4$

$p_6 = 1 - 25x^2 + 625x^4 - 15625x^6$

Note

$\displaystyle\int_{-1}^{1} f(x)\,dx$ is poorly approximated by $\displaystyle\int_{-1}^{1} p_n(x)\,dx$.

$\displaystyle \frac{1}{1+25x^2}$