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Lecture 24

Coefficient matrix is lower triangular and invertible if $x_0, x_1, \dots, x_n$ are distinct points. The operation count for finding coefficients $a_0, a_1, \dots, a_n$ using the table of divided differences is $\frac{n^2}{2}$. This is the same as forward elimination, but with the divided difference table, we don't need to create matrix elements.

<u>Thm</u> (<u>Error in polynomial interpolation</u>)

Let f be defined on $[a, b]$ and $f^{(n+1)}(x)$ be continuous on $[a, b]$. Let $x_0, x_1, \dots, x_n$ be $n+1$ distinct points on $[a, b]$, $p_n(x)$ be an interpolating polynomial of deg $\le n$ that interpolates $f(x)$ at $x_0, x_1, \dots, x_n$. Given $x \in [a, b]$, there exists a point $\xi \in (a, b)$ such that

$$\underset{\text{exact}}{f(x)} = \underset{\text{approximation}}{p_n(x)} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n)}_{\text{error}}$$

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Note This resembles Taylor Theorem.

Pf

~~If~~ $x = x_i$, $i = 0, 1, \dots, n$, then any ξ would work.

Let $x \neq x_i$, $i = 0, 1, \dots, n$.

~~Let~~ $g(t)$ be an interpolating polynomial of deg $\le n+1$ that interpolates $f(t)$ at points $t = x_0, x_1, \dots, x_n, x$: $n+2$ distinct points.

$$g(t) = p_n(t) + f[x_0, x_1, \dots, x_n, x](t - x_0)(t - x_1)\dots(t - x_n)$$

(*) $$g(x) = p_n(x) + f[x_0, x_1, \dots, x_n, x](x - x_0)(x - x_1)\dots(x - x_n) = f(x)$$

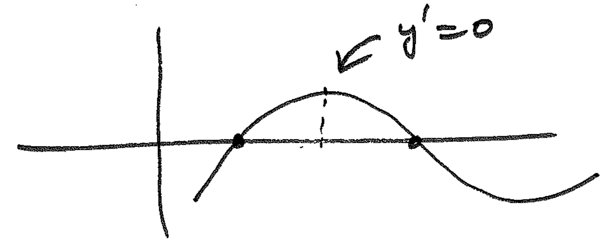
since $g(t)$ interpolates $f(t)$ at $x_0, x_1, \dots, x_n, x$.

Define $e(t) = f(t) - g(t)$

$e(t) = 0$ for $t = x_0, x_1, \dots, x_n, x$: at least $n+2$ distinct roots in $[a, b]$.

$e(t)$ has (at least) $n+2$ distinct roots in $[a, b]$

by MVT,



$e'(t)$ has (at least) $n+1$ distinct roots in $[a,b]$

$e''(t)$ has (at least) n distinct roots in $[a,b]$

$e^{(n+1)}(t)$ has (at least) one roots in $[a,b]$, call it ξ

$$\Rightarrow \quad e^{(n+1)}(\xi) = 0$$

$$\Rightarrow \quad 0 = e^{(n+1)}(\xi) = f^{(n+1)}(\xi) - g^{(n+1)}(\xi)$$

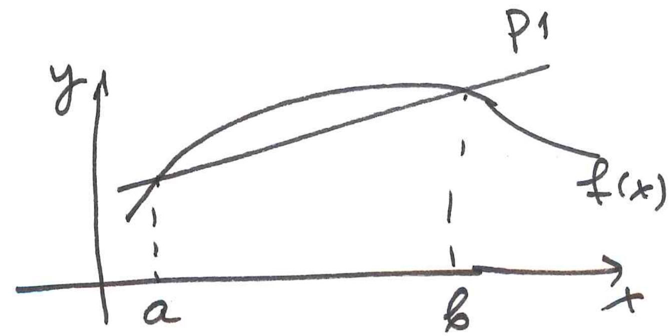
$$g^{(n+1)}(\xi) = \cancel{p_n^{(n+1)}(\xi)}^{\,0} + f[x_0, x_1, \dots, x_n, x] \cdot (n+1)!$$

$$\Rightarrow \quad f^{(n+1)}(\xi) - \underbrace{f[x_0, x_1, \dots, x_n, x] \cdot (n+1)!}_{g^{(n+1)}(\xi)} = 0$$

$$\Rightarrow \quad f[x_0, x_1, \dots, x_n, x] = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

Substitute $f[x_0, x_1, \ldots, x_n, x]$ into ⊛ to get

$$\boxed{f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\ldots(x-x_n)}$$



Application: $n=1$, $x_0 = a$, $x_1 = b$

Claim

If $|f''(x)| \le M$ for all $x \in [a,b]$, then

$$\|f(x) - p_1(x)\|_\infty = \sup_{x\in[a,b]} |f(x) - p_1(x)| \le \frac{M}{8}(b-a)^2$$

Pf

$$|f(x) - p_1(x)| = \left| \frac{f''(\xi)}{2!}(x-a)(x-b) \right| \le \frac{M}{2}|(x-a)(x-b)|$$

$$g(x) = (x-a)(x-b) = x^2 - (a+b)x + ab$$

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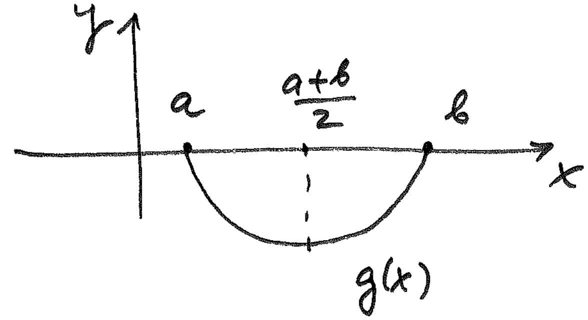
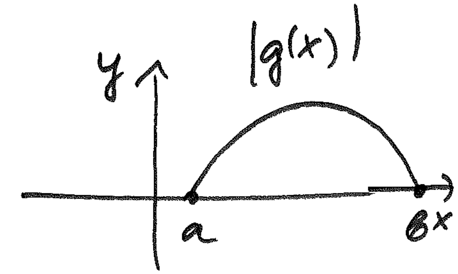
$g'(x) = 2x - (a+b)$ $\quad$ $g'(x) = 0$ at $\bar{x} = \frac{a+b}{2}$: critical point

$g''(x) = 2 > 0 \Rightarrow g(x)$ has local min at $x = \bar{x}$

$g(a) = 0, \quad g(b) = 0$





$\Rightarrow |g(x)|$ has local max (and global max) at $x = \bar{x}$

$$|g(\bar{x})| = \left|\left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)\right| = \left|\frac{b-a}{2} \cdot \frac{a-b}{2}\right| = \frac{(b-a)^2}{4}$$

$$\Rightarrow |g(x)| \le |g(\bar{x})| = \frac{(b-a)^2}{4}$$

$$\Rightarrow |f(x) - p_1(x)| \le \frac{M}{2} |(x-a)(x-b)| \le \frac{M}{2} \cdot \frac{(b-a)^2}{4} = \frac{M}{8}(b-a)^2$$

true for any $x \in [a,b]$

$$\Rightarrow \|f - p_1\|_\infty = \sup_{[a,b]} |f - p_1| \le \frac{M}{8}(b-a)^2$$

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Ex $f(x) = \frac{1}{x}$, $x_0 = a = 1$, $x_1 = b = 2$

$f' = -\frac{1}{x^2}$, $f'' = \frac{2}{x^3}$

$M = \sup_{1 \le x \le 2} |f''(x)| = \max_{1 \le x \le 2} |f''(x)| = \frac{2}{1^3} = 2 \Rightarrow M = 2$

$\|f - p_1\|_\infty \le \frac{M}{8}(b-a)^2 = \frac{2}{8}(2-1)^2 = \frac{1}{4}$

Questions

1. Given f, $[-1, 1]$, what is the best choice of interpolating points $x_0, x_1, \dots, x_n$?
2. $p_n \to f$ for all $x \in [a, b]$ as $n \to \infty$?

Recall

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \underbrace{(x-x_0)(x-x_1)\cdots(x-x_n)}_{\omega(x)}$$

Ex $$f(x) = \frac{1}{1+25x^2} = \frac{1}{1-(5ix)^2} = \frac{1}{(1-5ix)(1+5ix)} =$$

use partial fraction decomposition

$$= \frac{1}{2}\left(\frac{1}{1-5ix} + \frac{1}{1+5ix}\right)$$

$$f^{(n+1)}(x) = \frac{1}{2}(-1)^{n+1}(n+1)!\left(\frac{(-5i)^{n+1}}{(1-5ix)^{n+2}} + \frac{(5i)^{n+1}}{(1+5ix)^{n+2}}\right)$$