University of Idaho

Lecture 26

Claim Let $l_i(x)$ be Lagrange polynomials

$$l_i(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Then

$$h_j(x) = (1 - 2(x - x_j)\, l_j'(x_j))\, l_j^2(x)$$

$$\tilde{h}_j(x) = (x - x_j)\, l_j^2(x)$$

Check

$$h_j(x_i) = (1 - 2(x_i - x_j)\, l_j'(x_j))\, l_j^2(x_i) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$h_j'(x) = -2\, l_j'(x_j) \cdot l_j^2(x) + (1 - 2(x - x_j)\, l_j'(x_j)) \cdot 2\, l_j(x)\, l_j'(x)$$

$$h_j'(x_i) = -2 l_j'(x_j) \cdot l_j^2(x_i) + (1 - 2(x_i - x_j)\, l_j'(x_j)) \cdot 2\, l_j(x_i)\, l_j'(x_i)$$

$$= \begin{cases} 0 & i \neq j \\ 0 & i = j \end{cases}$$

$$\tilde{h}_j(x_i) = (x_i - x_j)\, \ell_j^2(x_i) = \begin{cases} 0 & i \neq j \\ 0 & i = j \end{cases}$$

$$\tilde{h}_j'(x) = \ell_j^2(x) + (x - x_j) \cdot 2\ell_j(x) \cdot \ell_j'(x)$$

$$\tilde{h}_j'(x_i) = \ell_j^2(x_i) + (x_i - x_j) \cdot 2\ell_j(x_i)\, \ell_j'(x_i) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Newton's form $\{x_0, x_1\}$, $f(x)$

z_0	x_0	$f[x_0]$	$f'(x_0)$	$f[z_0, z_1, z_2]$	$f[z_0, z_1, z_2, z_3]$
z_1	x_0	$f[x_0]$	$f[x_0, x_1]$	$f[z_1, z_2, z_3]$	
z_2	x_1	$f[x_1]$	$f'(x_1)$		
z_3	x_1	$f[x_1]$			

$$p(x) = f[x_0] + f'(x_0)(\underset{\overset{\parallel}{x - x_0}}{x - z_0}) + f[z_0, z_1, z_2)\overbrace{(x - z_0)(x - z_1)}^{\overset{(x-x_0)^2}{\parallel}} +$$

$$+ f[z_0, z_1, z_2, z_3]\underbrace{(x - z_0)(x - z_1)(x - z_2)}_{(x - x_0)^2 (x - x_1)}$$

$$\begin{array}{ll} x_0 & f[x_0] \\ x_0+\varepsilon & f[x_0+\varepsilon] \end{array} \qquad \frac{f[x_0+\varepsilon]-f[x_0]}{(x_0+\varepsilon)-x_0} = \frac{f(x_0+\varepsilon)-f(x_0)}{\varepsilon} \xrightarrow[\varepsilon\to 0]{} f'(x_0)$$

---

x_0

$x_0+\varepsilon$

$x_0+2\varepsilon$

$$\begin{array}{llll} x_0 & f[x_0] & & \\ & & f'(x_0) & \\ x_0 & f[x_0] & & \frac{1}{2!}f''(x_0) \\ & & f'(x_0) & \\ x_0 & f[x_0] & & \end{array}$$

Ex $x_0 = 0, \quad f_0 = 1, \quad f_0' = 0$

$x_1 = 1, \quad f_1 = 0, \quad f_1' = 0$

$$
\begin{array}{llllll}
z_0 & 0 & 1 & \overset{f_0'}{0} & & \\
z_1 & 0 & 1 & \frac{0-1}{1-0} = -1 & \frac{-1-0}{1-0} = -1 & \\
z_2 & 1 & 0 & \underset{f_1'}{0} & \frac{0-(-1)}{1-0} = 1 & \frac{1-(-1)}{1-0} = 2 \\
z_3 & 1 & 0 & & &
\end{array}
$$

$$p(x) = 1 + 0 \cdot \overset{x-z_0}{(x-0)} + (-1)\overset{(x-z_0)(x-z_1)}{(x-0)(x-0)} + 2\overset{(x-z_0)(x-z_1)(x-z_2)}{(x-0)(x-0)(x-1)} =$$

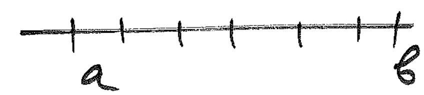
$$= \underbrace{1 - x^2}_{(1-x)(1+x)} + 2x^2(x-1) = (x-1)\left(-1-x+2x^2\right) = +(x-1)\left(2x^2 - x - 1\right) =$$

$$= (x-1)^2(2x+1) = h_0(x)$$

$$p(x) = \underbrace{f_0}_{=1} \cdot h_0(x) + \cancelto{0}{f_1} \cdot h_1(x) + \cancelto{0}{f_0'} \cdot \tilde{h}_0(x) + \cancelto{0}{f_1'} \cdot \tilde{h}_1(x) = h_0(x)$$

Piecewise polynomial interpolation

Given function f, $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$



The interpolating polynomial of degree $\le n$, p_n, may not be a good approximation of $f(x)$ over the entire interval $[a, b]$.

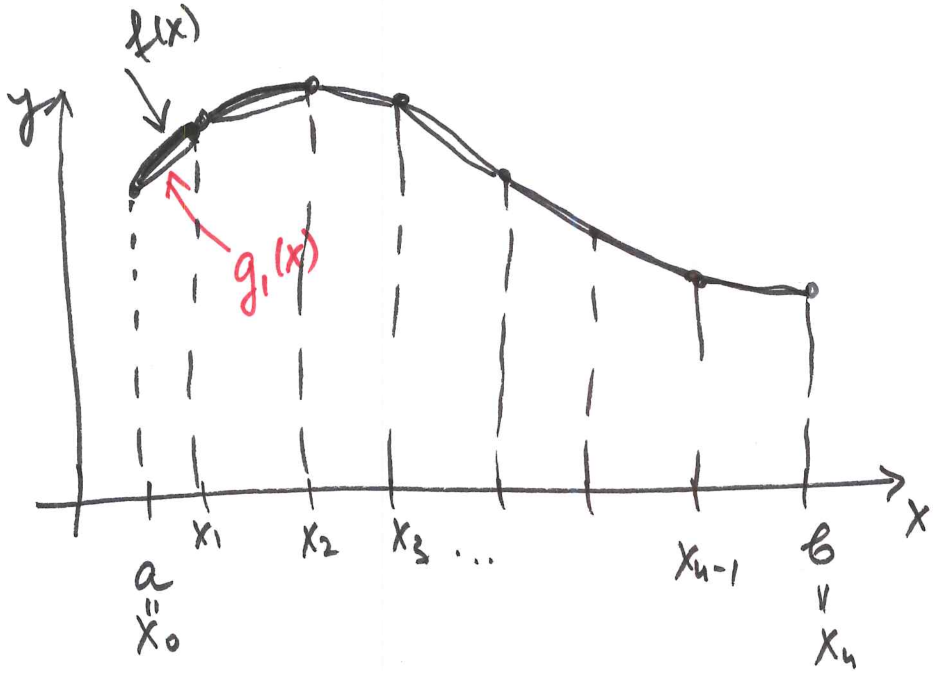
Define g_1: piecewise linear interpolant by

$$g_1(x) = f[x_i] + f[x_i, x_{i+1}](x - x_i), \quad x \in [x_i, x_{i+1}]$$

$$g_1(x_i) = f[x_i] = f(x_i)$$

$$g_1(x_{i+1}) = f[x_i] + f[x_i, x_{i+1}](x_{i+1} - x_i) =$$

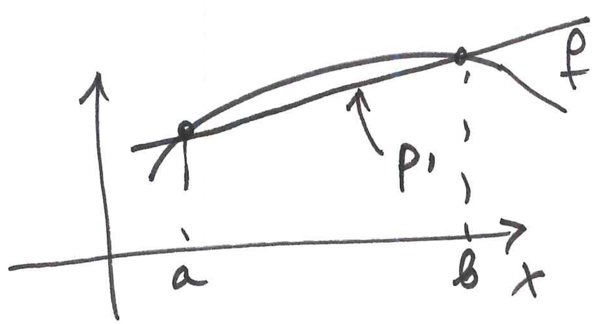
$$= f[x_i] + \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}(x_{i+1} - x_i) = f(x_{i+1})$$



$g_1(x)$ is continuous on $[a, b]$ but not differentiable on $[a, b]$ (i.e. $g_1(x)$ is not differentiable at x_i, $i = 1, \dots, n-1$)

Let $M_i = \max\{|f''(x)|,\ x \in [x_i, x_{i+1}]\}$

Recall



We showed earlier:

$$\|f - p_1\|_\infty \le \frac{M}{8}(b-a)^2$$

$$M = \sup_{[a,b]} |f''(x)|$$

For any $x \in [a, b]$

$$|f(x) - g_1(x)| \leq \max_i \left\{ \frac{M_i}{8} |x_{i+1} - x_i|^2 \right\}$$