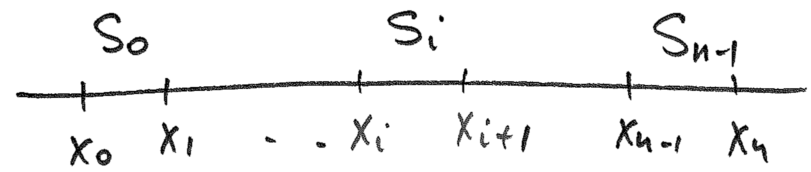
Lecture 28



__step 4__ (boundary conditions)

$S_0''(x_0) = 0 \Rightarrow a_0 = 0$

$S_{n-1}''(x_n) = 0 \Rightarrow a_n = 0$

(This will general natural spline)

$S_i''(x_i) = a_i$

$S_i''(x_{i+1}) = a_{i+1}$

or $S_{i-1}''(x_i) = a_i$

$$\begin{pmatrix} 4 & 1 & & & 0 \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ 0 & & & 1 & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-2} \\ a_{n-1} \end{pmatrix} = \frac{6}{h^2} \begin{pmatrix} f_0 - 2f_1 + f_2 \\ \vdots \\ f_{n-2} - 2f_{n-1} + f_n \end{pmatrix}$$

Coefficient matrix is symmetric, tridiagonal, strictly diagonally dominant, positive definite

$\Rightarrow$ The above system $(*)$ has a solution

Recall

$A=(a_{ij})$ is strictly diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|, \qquad i=1,2,\dots,n$$

<u>Thm</u> Strictly diagonally dominant matrix is invertible.

In practice, coefficients $a_1, a_2, \dots, a_{n-1}$ are computed using Gaussian elimination of a tridiagonal matrix.

<u>Note</u> Clamped BCs: $S_0'(x_0) = f'(x_0)$

$S_{n-1}'(x_n) = f'(x_n)$

Coefficients a_0 and a_1 are not zero anymore.

$$S_0'(x) = -\frac{a_0}{2h}(x_1 - x)^2 + \frac{a_1}{2h}(x - x_0)^2 - \left(\frac{f_0}{h} - \frac{a_0 h}{6}\right) + \left(\frac{f_1}{h} - \frac{a_1 h}{6}\right)$$

$$S_0'(x_0) = -\frac{a_0}{2}h - \left(\frac{f_0}{h} - \frac{a_0 h}{6}\right) + \left(\frac{f_1}{h} - \frac{a_1 h}{6}\right) = f_0' \qquad (**_1)$$

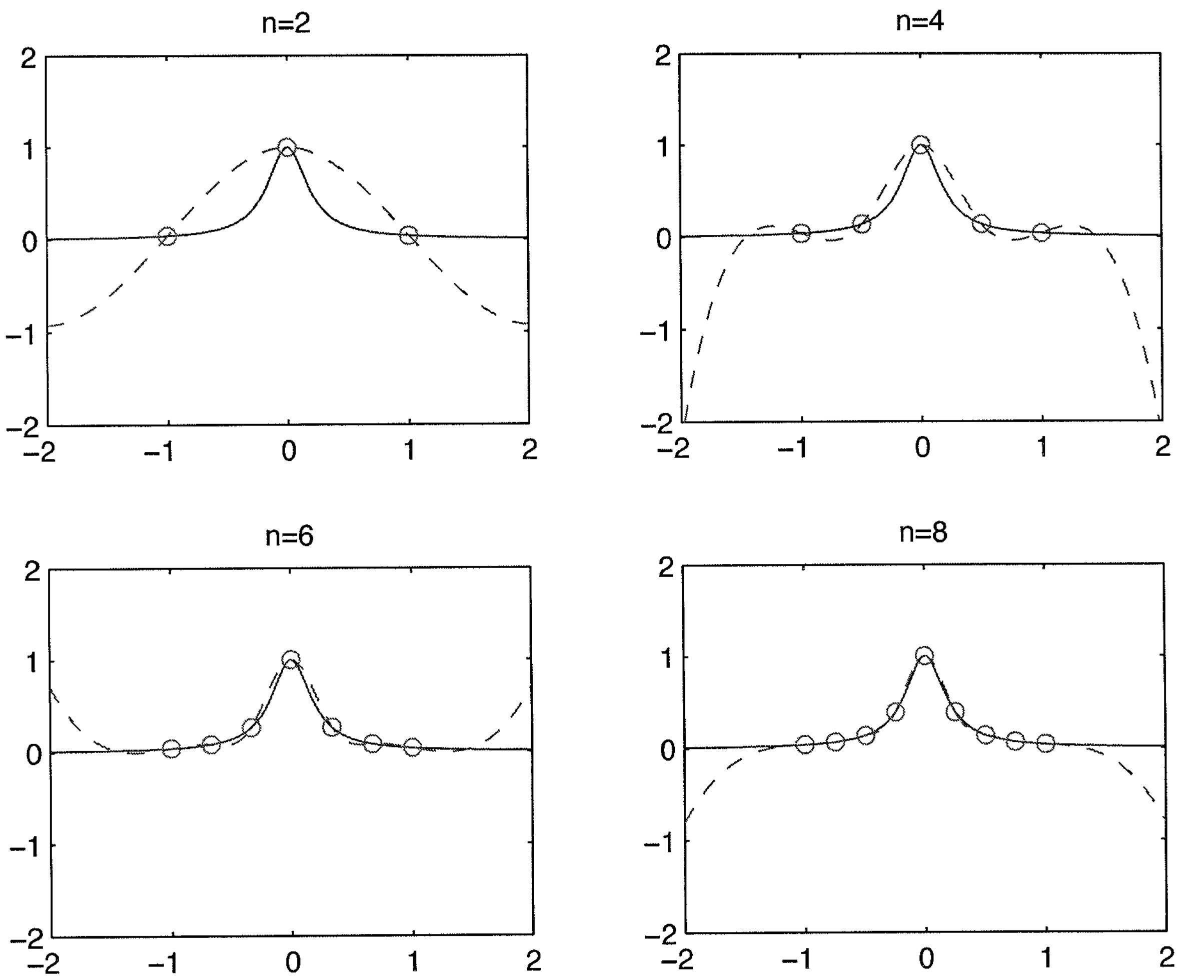
gives an extra equation for a_0 and a_1.

Similarly, condition $S_{n-1}'(x_n) = f_n'$ $(**_2)$ gives another eq$^{\underline{n}}$ for a_{n-1} and a_n.

equations $(**_1)$, $(**_2)$ together w/ $(*)$ from the previous lecture will form a system of $n+1$ unknowns $a_0, a_1, \dots, a_n$.

<u>ex</u> : natural cubic spline interpolation

$$f(x) = \frac{1}{1+25x^2}\ , \quad -1 \le x \le 1\ ,\ x_i = -1 + ih\ ,\ h = \frac{2}{n}\ ,\ i = 0 : n$$



<u>note</u>

1. $|f(x) - s(x)| \le \frac{5}{384} \max_{a \le x \le b} |f^{(4)}(x)|\, h^4$: 4th order accurate , <u>pf</u> : omit

2. The natural cubic spline interpolant has inflection points at the endpoints of the interval, due to the boundary conditions $s''(x_0) = s''(x_n) = 0$. In fact, there are additional inflection points in the interior of the interval, which are problematic in some applications.

Thm

Let $f(x)$ be defined on $[a, b]$, $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ and let $S(x)$ be a cubic spline interpolant of f with natural or clamped boundary conditions.

1. $\displaystyle |f(x) - S(x)| \le \frac{5}{384} \max_{a \le x \le b} |f^{(4)}(x)| \cdot h^4$

where $h = \max_i |x_{i+1} - x_i|$.

2. $\displaystyle \int_a^b [S''(x)]^2 dx \le \int_a^b [f''(x)]^2 dx$

The first condition says that spline interpolation is 4th order accurate.

Recall

$$\underset{\text{curvature}}{K(x)} = \frac{|f''(x)|}{\left(1 + [f'(x)]^2\right)^{3/2}} \approx |f''(x)|$$

$\int_a^b [f''(x)]^2 dx$ is a crude measure of the total curvature over $[a,b]$.

The second result can be interpreted as optimality property or minimal curvature property. It means if one considers any other interpolant, it will oscillate at least as much as a spline (natural spline: $S''(x_0) = S''(x_n) = 0$ or clamped spline: $S'(x_0) = f'(x_0)$, $S'(x_n) = f'(x_n)$).

Numerical Integration

Newton - Côtes formulas

Basic idea

$$\int_a^b f(x)\,dx \sim \sum_{i=0}^{n} c_i f(x_i)$$

For now, assume $x_i = a + ih$, $h = \frac{b-a}{n}$

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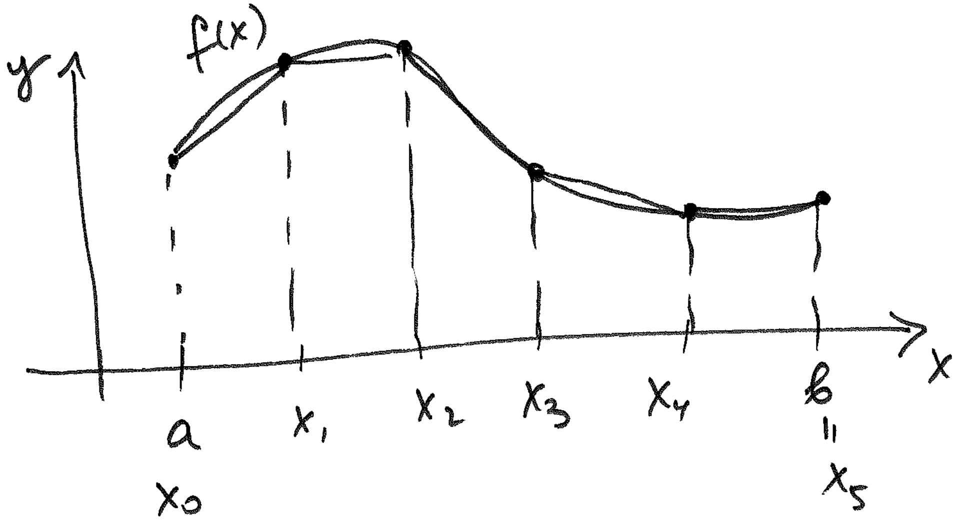
Trapezoid Rule



$$T(h) = h\,\frac{f(x_0)+f(x_1)}{2} + \dots + h\,\frac{f(x_{n-1})+f(x_n)}{2}$$

$$\boxed{T(h) = h\left(\frac{1}{2}f(x_0) + f(x_1) + \dots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right)}$$

Trapezoid Rule

Ex $\int_0^1 e^{-x^2}\,dx = 0.746824\ldots$

h	$T(h)$	error	error$/h^2$
1	0.683940	0.062884	0.062884
0.5	0.731370	0.015454	0.061816
0.25	0.742984	0.003840	0.06144
0.125	0.745866	0.000958	0.061312

This implies that Trapezoid Rule is 2^{nd} order accurate.
We want to prove this analytically.

<u>Local error analysis</u> (Trapezoid Rule)