Lecture 30

Recall

$$\int_0^h f(x)\,dx \sim c_0 f(0) + c_1 f(h) \qquad (*)$$

$\left.\begin{array}{l} f(x)=1 \\ f(x)=x \end{array}\right\}$ we required that $\int_0^h f(x)\,dx = c_0 f(0) + c_1 f(h)$ for $f(x)=1$ and $f(x)=x$, i.e.

formula $(*)$ is exact for polynomials of deg ≤ 1.

$$\Rightarrow c_0 = c_1 = \frac{h}{2}$$

$$\Rightarrow \int_0^h f(x)\,dx = h \cdot \frac{f(0)+f(h)}{2} = \frac{h}{2} f(0) + \frac{h}{2} f(h)$$

Note

$f(x) = x^2$

$$\int_0^h x^2\,dx = \frac{x^3}{3}\Big|_0^h = \frac{h^3}{3} \overset{?}{\neq} \frac{h}{2}\cdot 0 + \frac{h}{2}\cdot h^2 = \frac{h^3}{2}$$

$$\Rightarrow \int_0^h f(x)\,dx \sim \underset{\frac{h}{2}}{\underset{\shortparallel}{c_0}} f(0) + \underset{\frac{h}{2}}{\underset{\shortparallel}{c_1}} f(h)$$ is not exact for polynomials of degree 2

Thus, the trapezoid rule is exact for polynomials of degree ≤ 1. Trapezoid rule has degree of precision $r = 1$.

Asymptotic Expansion

$$4. \quad T(h) = \int_a^b f(x)\,dx + c_2 h^2 + c_4 h^4 + c_6 h^6 + \dots$$

since error $\sim c h^2$ (cumulative error)

Richardson extrapolation (Romberg's method)

$$T(2h) = \int_a^b f(x)\,dx + c_2 (2h)^2 + c_4 (2h)^4 + c_6 (2h)^6 + \dots$$

$$T(2h) = \int_a^b f(x)\,dx + 4c_2 h^2 + 16 c_4 h^4 + 64 c_6 h^6 + \dots$$

Multiply expression with $T(h)$ by 4 and subtract $T(2h)$ from $4T(h)$:

$$4T(h) - T(2h) = 3\int_a^b f(x)\,dx - 12C_4h^4 - 60C_6h^6 - \dots$$

$$\frac{4T(h) - T(2h)}{3} = \int_a^b f(x)\,dx - 4C_4h^4 - 20C_6h^6 - \dots$$

Define $R_1(h) = \dfrac{4T(h) - T(2h)}{3}$: $\underline{4^{th} \text{ order accurate}}$

$$16\cdot \Big|\; R_1(h) = \int_a^b f(x)\,dx + \tilde{C}_4h^4 + \tilde{C}_6h^6 + \dots$$

$$- \quad R_1(2h) = \int_a^b f(x)\,dx + \tilde{C}_4\underbrace{(2h)^4}_{=16h^4} + \tilde{C}_6(2h)^6 + \dots$$

$$\frac{16R_1(h) - R_1(2h)}{15} = \int_a^b f(x)\,dx - \frac{16}{5}\tilde{C}_6h^6 - \dots$$

Define $R_2(h) = \dfrac{16R_1(h) - R_1(2h)}{15}$: $\underline{6^{th} \text{ order accurate}}$

<u>Note</u>

Let $R_0(h) = T(h)$, then

$$R_1(h) = \frac{4R_0(h) - R_0(2h)}{3} = R_0(h) + \frac{R_0(h) - R_0(2h)}{3}$$

$$R_2(h) = \frac{16R_1(h) - R_1(2h)}{15} = R_1(h) + \frac{R_1(h) - R_1(2h)}{15}$$

$$R_3(h) = \frac{64R_2(h) - R_2(2h)}{63} = R_2(h) + \frac{R_2(h) - R_2(2h)}{63}$$

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$$R_n(h) = \frac{4^n R_{n-1}(h) - R_{n-1}(2h)}{4^n - 1}$$

Table

$2h$	$R_0(2h)$					
		$\frac{\Delta}{3}$				
h	$R_0(h)$		$R_1(h)$			
		$\frac{\Delta}{3}$		$\frac{\Delta}{15}$		
$\frac{h}{2}$	$R_0(\frac{h}{2})$		$R_1(\frac{h}{2})$		$R_2(\frac{h}{2})$	

Note

down a column: decreasing h, fixed order of accuracy

across a row: fixed h, increasing order of accuracy

Ex $\displaystyle\int_0^1 e^{-x^2}\,dx = 0.74682413\ldots$

h	$R_0(h)$ Trapezoid rule	$R_1(h)$ Simpson's rule	$R_2(h)$ Boole's rule	$R_3(h)$
1	0.683940			
0.5	0.731370	0.747180		
0.25	0.742984	0.7468553	0.7468336	
0.125	0.745866	0.7468266	0.7468246	0.7468244

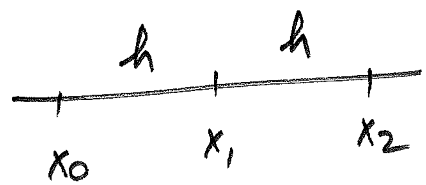
Note Value in the last column ($R_3(0.125)$) is obtained by using a difference divided by 63, i.e. $\frac{\Delta}{63}$.

Simpson's Rule

$$R_1(h) = R_0(h) + \frac{R_0(h) - R_0(2h)}{3}$$

or $$R_1(h) = T(h) + \frac{T(h) - T(2h)}{3}$$

local form



$$T(h) = h\left(\tfrac{1}{2}f(x_0) + f(x_1) + \tfrac{1}{2}f(x_2)\right)$$

$$T(2h) = 2h\left(\tfrac{1}{2}f(x_0) + \tfrac{1}{2}f(x_2)\right) = h\left(f(x_0) + f(x_2)\right)$$

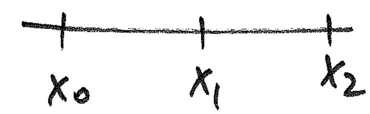
$$R_1(h) = h\left(\tfrac{1}{2}f(x_0) + f(x_1) + \tfrac{1}{2}f(x_2)\right) +$$

$$+\frac{1}{3}\Bigg(\underbrace{\frac{h}{2}f(x_0)+h f(x_1)+\frac{h}{2}f(x_2)}_{T(h)}-\underbrace{h f(x_0)-h f(x_2)}_{T(2h)}\Bigg)=$$

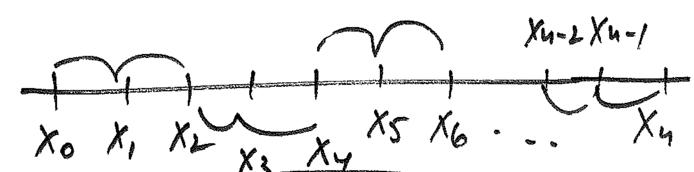
$$= f(x_0)\left(\frac{h}{2}+\frac{h}{6}-\frac{h}{3}\right)+f(x_1)\left(h+\frac{h}{3}\right)+f(x_2)\left(\frac{h}{2}+\frac{h}{6}-\frac{h}{3}\right)=$$

$$=\frac{h}{3}f(x_0)+\frac{4h}{3}f(x_1)+\frac{h}{3}f(x_2)$$

$$\Rightarrow \boxed{R_1(h)=\frac{h}{3}f(x_0)+\frac{4h}{3}f(x_1)+\frac{h}{3}f(x_2)}$$



Global form (assume n is even)



$$\boxed{\begin{aligned}R_1(h)=h\Big(&\frac{1}{3}f(x_0)+\frac{4}{3}f(x_1)+\frac{2}{3}f(x_2)+\frac{4}{3}f(x_3)+\frac{2}{3}f(x_4)+\dots\\&+\frac{2}{3}f(x_{n-2})+\frac{4}{3}f(x_{n-1})+\frac{1}{3}f(x_n)\Big)\end{aligned}}$$

Simpson's Rule

Error

$$\int_a^b f(x)\,dx = R_1(h) - \frac{f^{(4)}(\hat{\xi})}{180} h^4 (b-a), \qquad \hat{\xi} \in [a,b]$$

Ex Use the method of undetermined coefficients to derive Simpson's rule.

$$\int_0^{2h} f(x)\,dx \sim c_0 f(0) + c_1 f(h) + c_2 f(2h) \qquad x_0 = 0, \; x_1 = h, \; x_2 = 2h$$

$f(x) = 1$ $\qquad \int_0^{2h} 1\,dx = \boxed{2h = c_0 \cdot 1 + c_1 \cdot 1 + c_2 \cdot 1}$ ✓

$f(x) = x$ $\qquad \int_0^{2h} x\,dx = \frac{x^2}{2}\Big|_0^{2h} = \boxed{2h^2 = c_0 \cdot 0 + c_1 \cdot h + c_2 \cdot 2h}$

$\Rightarrow \boxed{2h = c_1 + 2c_2}$ ✓

$f(x) = x^2$ $\qquad \int_0^{2h} x^2\,dx = \frac{x^3}{3}\Big|_0^{2h} = \boxed{\frac{8}{3}h^3 = c_0 \cdot 0 + c_1 \cdot h^2 + c_2 (2h)^2}$

$\Rightarrow \boxed{\frac{8}{3}h = c_1 + 4c_2}$ ✓

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2h \\ 2h \\ 8h/3 \end{pmatrix}$$

Solution: $\boxed{c_0 = c_2 = \frac{h}{3}, \quad c_1 = \frac{4h}{3}}$