Lecture 31

Note

$f(x)=x^3$

$$\int_0^{2h} x^3\,dx = \frac{x^4}{4}\Big|_0^{2h} = 4h^4 \stackrel{?}{=} \underbrace{c_0}_{=\frac{h}{3}} f(0) + \underbrace{c_1}_{=\frac{4h}{3}} f(h) + \underbrace{c_2}_{=\frac{h}{3}} f(2h)$$

$$\stackrel{?}{=} \frac{h}{3}\cdot \cancel{0} + \frac{4h}{3}\cdot h^3 + \frac{h}{3}\cdot(2h)^3$$

$$\stackrel{?}{=} \left(\frac{4}{3}+\frac{8}{3}\right)h^3 = 4h^3 \quad \checkmark \quad =$$

$$\Rightarrow \int_0^{2h} f(x)\,dx = \frac{h}{3} f(0) + \frac{4h}{3} f(h) + \frac{h}{3} f(2h)$$

i.e. Simpson's rule is exact for polynomials of deg ≤ 3 at least.

$f(x)=x^4$

$$\int_0^{2h} x^4\,dx \neq \frac{h}{3}\cdot 0 + \frac{4h}{3} h^4 + \frac{h}{3}(2h)^4$$

Hence, Simpson's rule is exact for polynomials of degree ≤ 3,

i.e. degree of precision for Simpson's rule is $r=3$.

Trapezoid and Simpson's rule are examples of Newton-Côtes formulas.

Orthogonal Polynomials

Define the inner product of two functions on $[-1, 1]$ as

$$\int_{-1}^{1} f(x)\, g(x)\, dx = \langle f, g \rangle$$

Properties

1. $\langle f, f \rangle \geq 0$ and $\langle f, f \rangle = 0 \Leftrightarrow f \equiv 0$

$$\langle f, f \rangle = \| f \|^2$$

$$\| f \| = \sqrt{\langle f, f \rangle} \ : \text{ norm of } f$$

2. $\langle f, \alpha h + g \rangle = \alpha \langle f, h \rangle + \langle f, g \rangle$

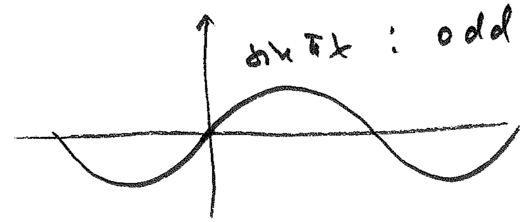
We say that functions f and g are orthogonal if $\langle f, g \rangle = 0$

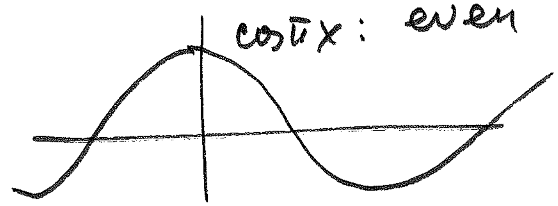
Ex $\sin \pi x$ and $\cos \pi x$ are orthogonal on $[-1, 1]$

$\langle \sin \pi x, \cos \pi x \rangle =$

$$= \int_{-1}^{1} \underbrace{\underset{\text{odd}}{\sin \pi x} \cdot \underset{\text{even}}{\cos \pi x}}_{\text{odd}}\, dx = \frac{1}{2} \int_{-1}^{1} \sin 2\pi x \, dx = -\frac{1}{2} \cos 2\pi x \cdot \frac{1}{2\pi} \Big|_{-1}^{1} = 0$$

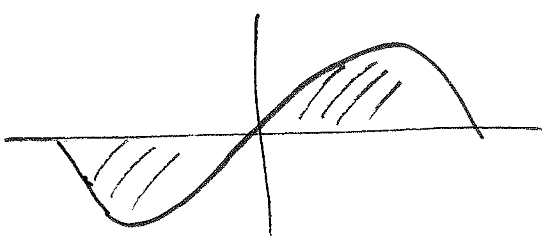
$\sin 2t = 2 \sin t \cdot \cos t$



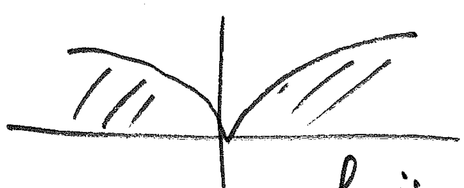


$f(-x) = -f(x)$: f odd

$f(-x) = f(x)$: f is even



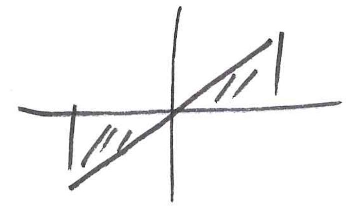
$$\Rightarrow \int_{-1}^{1} \underset{\text{odd}}{f(x)}\, dx = 0$$



f is even

$$\int_{-1}^{1} \underset{\text{even}}{f(x)}\, dx = 2 \int_{0}^{1} f(x)\, dx$$

Ex $\langle 1, x\rangle = \int_{-1}^{1} 1\cdot x\,dx = 0$: $\Rightarrow$ 1 and x are orthogonal on $[-1,1]$



$$\langle 1, x^2\rangle = \int_{-1}^{1} \underbrace{1\cdot \underbrace{x^2}_{\text{even}}}_{\text{even}}\,dx = 2\int_0^1 x^2\,dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3} \neq 0$$

$\Rightarrow$ 1 and x^2 are not orthogonal

Gram-Schmidt orthogonaligation method

Given a sequence of linearly independent functions $\{\varphi_0, \varphi_1, \varphi_2, \dots\}$, the Gram-Schmidt orthogalization method produces a sequence $\{\psi_0, \psi_1, \psi_2, \dots\}$ of mutually orthogonal functions.

In particular, given a sequence $\{1, x, x^2, x^3, \dots\}$: linearly independent

Gram-Schmidt process gives a sequence of orthogonal polynomials $\{p_0(x), p_1(x), p_2(x) \dots\}$ called <u>Legendre polynomials</u>.

$p_0 = 1$

$p_1 = x + \alpha_{10} \cdot p_0 \quad | \cdot p_0$

We want to find coefficient α_{10} such that p_1 and p_0 are orthogonal.

$$\overset{0}{\cancel{\langle p_1, p_0 \rangle}} = \langle x, p_0 \rangle + \alpha_{10} \underbrace{\langle p_0, p_0 \rangle}_{= \|p_0\|^2 \neq 0} \Rightarrow \alpha_{10} = -\frac{\langle x, p_0 \rangle}{\|p_0\|^2}$$

$$\left.\begin{aligned} &\langle x, p_0 \rangle = \langle x, 1 \rangle = 0 \\ &\|p_0\|^2 = \langle p_0, p_0 \rangle = \int_{-1}^{1} 1 \cdot 1 \, dx = 2 \end{aligned}\right\} \Rightarrow \alpha_{10} = 0 \Rightarrow p_1 = x + 0 \cdot p_0$$

$$\boxed{p_1 = x}$$

$p_2 = x^2 + \alpha_{21} \cdot p_1 + \alpha_{20} \cdot p_0 \quad | \cdot p_0 \quad | \cdot p_1$

We want to find α_{21}, α_{20} in such a way that p_2 is orthogonal to both p_0 and p_1.

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$$\overset{0}{\cancel{\langle p_2, p_0\rangle}} = \langle x^2, p_0\rangle + \alpha_{21}\overset{0}{\cancel{\langle p_1, p_0\rangle}} + \alpha_{20}\underbrace{\langle p_0, p_0\rangle}_{\|p_0\|^2} \Rightarrow \alpha_{20} = -\frac{\langle x^2, p_0\rangle}{\|p_0\|^2}$$

$$\left.\begin{array}{l}\langle x^2, p_0\rangle = \langle x^2, 1\rangle = \frac{2}{3} \\ \|p_0\|^2 = 2\end{array}\right\} \Rightarrow \boxed{\alpha_{20} = -\frac{2/3}{2} = -\frac{1}{3}}$$

$$\overset{0}{\cancel{\langle p_2, p_1\rangle}} = \langle x^2, p_1\rangle + \alpha_{21}\underbrace{\langle p_1, p_1\rangle}_{\substack{\|p_1\|^2 \\ \neq 0}} + \alpha_{20}\overset{0}{\cancel{\langle p_0, p_1\rangle}} \Rightarrow \alpha_{21} = -\frac{\langle x^2, p_1\rangle}{\|p_1\|^2}$$

$$\left.\begin{array}{l}\langle x^2, \underset{x}{p_1}\rangle = \int_{-1}^{1} x^2 \cdot x \, dx = \int_{-1}^{1} \underbrace{x^3}_{odd} dx = 0 \\ \|p_1\|^2 = \langle \underset{x}{p_1}, p_1\rangle = \int_{-1}^{1} x^2 dx = 2\int_0^1 x^2 dx = \frac{2}{3}\end{array}\right\} \Rightarrow \alpha_{21} = 0$$

$$\therefore \boxed{p_2 = x^2 - \frac{1}{3}}$$

Summary

$$p_0 = 1$$

$$p_1 = x + \alpha_{10} \cdot p_0 = x - \frac{\langle x, p_0 \rangle}{\|p_0\|^2} p_0 = x$$

$$p_2 = x^2 + \alpha_{21} p_1 + \alpha_{20} p_0 = x^2 - \frac{\langle x^2, p_1 \rangle}{\|p_1\|^2} p_1 - \frac{\langle x^2, p_0 \rangle}{\|p_0\|^2} p_0 = x^2 - \frac{1}{3}$$

Now

$$p_3 = x^3 - \cancel{\frac{\langle x^3, p_2 \rangle}{\|p_2\|^2}}^{\,0} p_2 - \frac{\langle x^3, p_1 \rangle}{\|p_1\|^2} \underbrace{p_1}_{=x} - \cancel{\frac{\langle x^3, p_0 \rangle}{\|p_0\|^2}}^{\,0} p_0$$

$$\langle x^3, p_1 \rangle = \int_{-1}^{1} x^3 \cdot x \, dx = \frac{2}{5}, \qquad \|p_1\|^2 = \frac{2}{3}$$

$$\Rightarrow \boxed{p_3 = x^3 - \frac{3}{5} x}$$

Note

1. $p_n(x)$ is a Legendre polynomial of degree n
2. Any polynomial of degree $\le n$ can be written

$$q(x) = \sum_{i=0}^{n} c_i p_i(x)$$

Legendre polynomials $\{p_i(x)\}_{i=0}^{n}$ form a basis of the set of polynomials $\mathcal{P}_n$ of degree $\le n$.

Gaussian Quadrature

1. Legendre polynomials $p_n(x)$ have n distinct roots in $(-1,1)$, say, x_i, $i=1,\dots,n$
2. There exist coefficients c_i, $i=1,\dots,n$ such that

$$\int_{-1}^{1} f(x)\,dx \sim \sum_{i=1}^{n} c_i f(x_i) = G_n$$

is <u>exact</u> for polynomials of degree $\leq 2n-1$.

<u>Ex</u>
$$\int_0^1 e^{-x^2}\,dx = \left| \begin{matrix} t = 2x-1 \\ dt = 2dx \end{matrix} \right| = \int_{-1}^{1} e^{-\frac{(t+1)^2}{4}} \cdot \frac{dt}{2} = \int_{-1}^{1} \underbrace{\frac{1}{2} e^{-\frac{(t+1)^2}{4}}}\,dt$$

n	G_n
2	0.746595
3	0.746816
4	0.746824

This is much better than with Trapezoid or Simpson's rules.