University of Idaho

Lecture 32

Alternative def of Legendre polynomials

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}\left[(1-x^2)^n\right], \qquad n \geq 1$$

$P_0(x) = 1$ $\qquad$ $\deg P_n = n$

$P_n(1) = 1$: normalization condition is used in this def

Claim

1) Legendre polynomial $P_n(x)$ has n distinct roots in $(-1,1)$, say, x_i, $i=1,\dots,n$

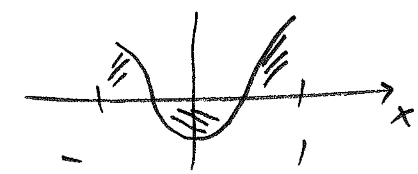
2) There exist constants c_i, $i=1,\dots,n$, such that

$$\int_{-1}^{1} f(x)\,dx \sim \sum_{i=1}^{n} c_i f(x_i) \qquad \underline{\text{Gauss quadrature}}$$

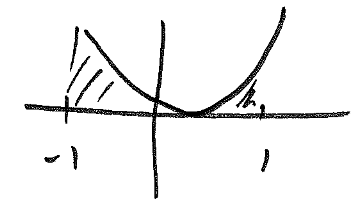
is exact for polynomials of degree $\leq 2n-1$, $n \geq 2$.

Pf of 1)

For $n \geq 1$, $\quad 0 = \langle P_n, P_0 \rangle = \int_{-1}^{1} P_n(x)\,dx$



This implies that $P_n(x)$ has at least one root (of odd multiplicity) in $(-1,1)$. Let x_i be points on $(-1,1)$ at which $P_n(x)$ changes its sign, $i = 1, \dots, j \leq n$.



$-1 < x_1 < x_2 < \dots < x_j < 1$ we know $\boxed{j \leq n}$

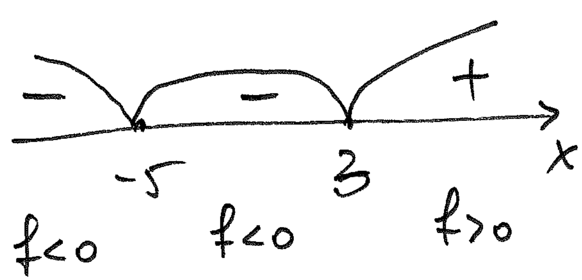
Form the function $g(x) = \prod_{i=1}^{j} (x - x_i) = (x-x_1)(x-x_2)\dots(x-x_j)$

Aside

Ex $f(x) = \overset{>0}{2}(x-3)(x+5)^2$

$x_1 = 3$, $m_1 = 1$

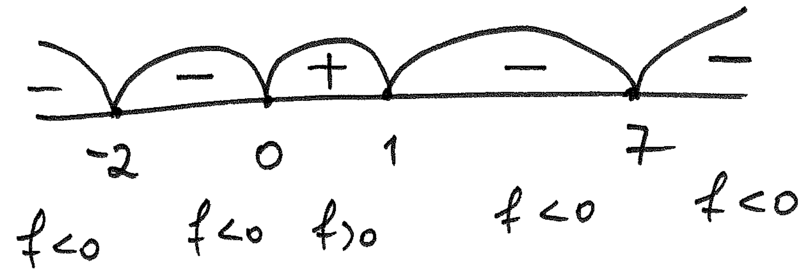
$x_2 = -5$, $m_2 = 2$



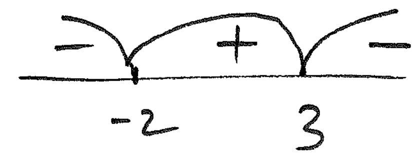
$P(x) = a(x-x_0)^{m_0}(x-x_1)^{m_1}\dots(x-x_k)^{m_k}$

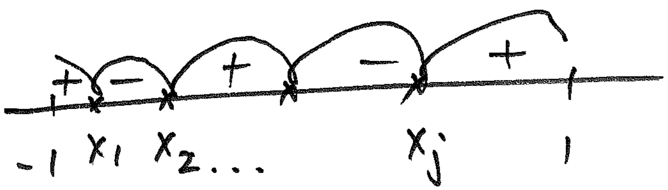
canonical form

Ex $f(x) = \underset{-1<0}{-}(x-1)(x+2)^4\, x^3\, (x-7)^2$



Ex $f(x) = (3-x)(x+2) = -1\,(x-3)(x+2)$





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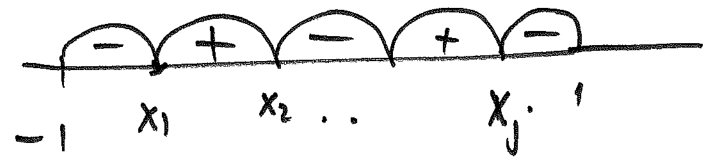
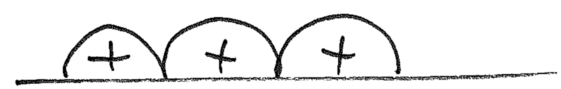


$f(x) = (x-1)$

$g(x) = (x-1)^3$

$\Rightarrow \quad f(x) = (x-1)^4$



or

sign of $g(x)$ also changes at points x_i, $i = 1, \dots, j$

Then $\langle P_n, g \rangle = \int_{-1}^{1} P_n(x)\, g(x)\, dx \neq 0$

since $P_n(x)$ and $g(x)$ have either positive or negative sign over each interval.

since $\langle P_n, g\rangle \neq 0 \Rightarrow P_n(x)$ and $g(x)$ are not orthogonal

But $\langle P_n, g\rangle \neq 0 \Rightarrow \overbrace{\deg g}^{=j} \geq n \Rightarrow \boxed{j \geq n}$

since otherwise if $\deg g < n \Rightarrow \langle P_n, g\rangle = 0$ since $\{P_0, P_1, \ldots, P_n\}$ form a basis in the set $\mathcal{P}_n$ of polynomials of degree $\leq n$.

i.e. if $\deg g < n \Rightarrow g = \alpha_0 P_0 + \alpha_1 P_1 + \ldots + \alpha_{n-1} P_{n-1} \quad \Big| \cdot P_n$

$\langle g, P_n\rangle = \alpha_0 \cancelto{0}{\langle P_0, P_n\rangle} + \alpha_1 \cancelto{0}{\langle P_1, P_n\rangle} + \ldots + \alpha_{n-1} \cancelto{0}{\langle P_{n-1}, P_n\rangle} \Rightarrow \langle g, P_n\rangle = 0$

Since $j \leq n$ and $j \geq n \Rightarrow \boxed{j = n}$, i.e. all roots of $P_n(x)$ are inside $(-1, 1)$.

<u>Note</u> all these roots are simple roots.

Proof 2

Assume that $f(x)$ is a polynomial of degree $\leq 2n-1$

Case 1 $\deg f \leq n-1$

$$f(x) = \sum_{j=1}^{n} f(x_j)\, l_j(x)$$

where $l_j(x)$ are Lagrange polynomials based on points $x_1, x_2, \dots, x_n$.

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \sum_{j=1}^{n} f(x_j)\, l_j(x)\,dx = \sum_{j=1}^{n} f(x_j) \underbrace{\int_{-1}^{1} l_j(x)\,dx}_{c_j} =$$

$$= \sum_{j=1}^{n} c_j f(x_j) \; : \text{ exact}$$

Note $c_j = \int_{-1}^{1} l_j(x)\,dx$

Case 2 $\deg f \leq 2n-1$

q : quotient, $\deg q \leq n-1$

r : remainder, $\deg r \leq n-1$

$f = q P_n + r$

$$\int_{-1}^{1} f(x)\,dx = \underbrace{\int_{-1}^{1} q(x) P_n(x)\,dx}_{\langle q, P_n \rangle \to 0} + \int_{-1}^{1} \underbrace{r(x)}_{\deg \leq n-1}\,dx \;=$$

use case 1

$\deg q < n$, $\deg P_n = n$

$$= \sum_{j=1}^{n} c_j \cdot r(x_j) \;\boxed{=}\; \text{(using case 1)}$$

$$f(x_j) = q(x_j)\,\cancel{P_n(x_j)}^{\,0} + r(x_j) = r(x_j)$$

as x_j are roots of P_n

$$\boxed{=} \sum_{j=1}^{n} c_j\, f(x_j) \;:\; \text{exact}$$

$\therefore \quad \int_{-1}^{1} f(x)\,dx = \sum_{j=1}^{n} c_j f(x_j)$ is exact for polynomials of deg $\leq 2n-1$

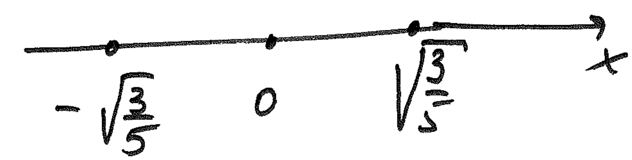
$$c_j = \int_{-1}^{1} \ell_j(x)\,dx$$

$\blacksquare$

<u>Ex</u> Derive 3-point Gauss quadrature rule.

$$\int_{-1}^{1} f(x)\,dx \sim c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

$P_3(x) = x^3 - \frac{3}{5}x = x\left(x^2 - \frac{3}{5}\right)$: $\underset{\text{3rd order}}{\wedge}$ Legendre polynomial



Take

$x_1 = -\sqrt{\frac{3}{5}}, \quad x_2 = 0, \quad x_3 = \sqrt{\frac{3}{5}}$

$c_j = \int_{-1}^{1} \ell_j(x)\,dx$, $\quad \ell_j(x)$: Lagrange polynomials

Another way to compute c_i is to use the method of undetermined coefficients.

$f(x)=1$

$$\int_{-1}^{1} 1\,dx = 2 = c_1 \underbrace{f(x_1)}_{=1} + c_2 \underbrace{f(x_2)}_{=1} + c_3 \underbrace{f(x_3)}_{=1}$$

$$\Rightarrow \boxed{2 = c_1 + c_2 + c_3}$$

$f(x)=x$

$$\int_{-1}^{1} x\,dx = 0 = \boxed{c_1\left(-\sqrt{\tfrac{3}{5}}\right) + c_2 \cdot 0 + c_3\sqrt{\tfrac{3}{5}}}$$

$f(x)=x^2$

$$\int_{-1}^{1} x^2\,dx = 2\int_{0}^{1} x^2\,dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3} =$$

$$= \boxed{c_1 \cdot \frac{3}{5} + c_2 \cdot 0 + c_3 \cdot \frac{3}{5} = \frac{2}{3}}$$

we find

$$c_1 = c_3 = \frac{5}{9}, \quad c_2 = \frac{8}{9}$$

3-point Gauss quadrature is

$$\int_{-1}^{1} f(x)\,dx \sim \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

$n=3 \Rightarrow 2n-1=5$

This formula is exact for polynomials of deg ≤ 5

Error in Gauss quadrature

$$\underbrace{\int_{-1}^{1} f(x)\,dx}_{\text{exact}} = \underbrace{\sum_{j=1}^{n} c_j\, f(x_j)}_{\text{approx.}} + \underbrace{E_n(f)}_{\text{error}}$$

where

$$E_n(f) = \frac{2^{2n+1}(n!)^4}{(2n+1)\left[(2n)!\right]^2} \frac{f^{(2n)}(\xi)}{(2n)!}$$

$n=2$: $E_2 f = \dfrac{f^{(4)}(\xi)}{135}$ $\qquad$ $n=3$ $\qquad$ $E_3 f = \dfrac{f^{(6)}(\xi)}{15,750}$