Lecture 33

Singular Integrals

Ex $\displaystyle\int_0^\infty f(x)e^{-x}\,dx$

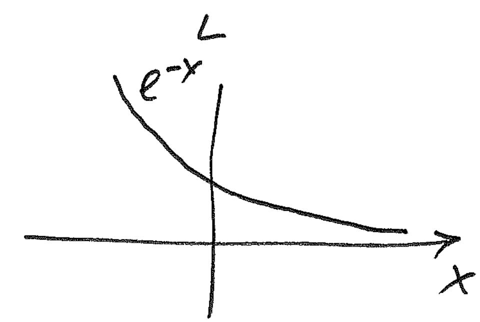
Method I (truncation of the domain)

$$\int_0^\infty f(x)e^{-x}\,dx = \underbrace{\int_0^L f(x)e^{-x}\,dx}_{1} + \underbrace{\int_L^\infty f(x)e^{-x}\,dx}_{2}$$

1 ~ use Trapezoid, Simpson's or Gauss quadrature

$$|2| = \left|\int_L^\infty f(x)\underbrace{e^{-x}}_{>0}\,dx\right| \le \int_L^\infty |f(x)|\cdot e^{-x}\,dx \le \|f\|_\infty \int_L^\infty e^{-x}\,dx =$$

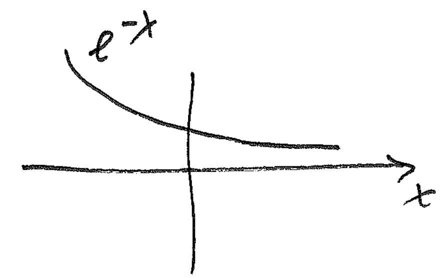
$$= \|f\|_\infty \cdot (-e^{-x})\Big|_{x=L}^{\infty} = \|f\|_\infty \cdot e^{-L}$$



Choose L large enough to make $\|f\|_\infty \cdot e^{-L}$ smaller than some given tolerance ε, then integrate (1), i.e.

$$\int_0^\infty f(x) e^{-x}\,dx \sim \int_0^L f(x) e^{-x}\,dx$$

<u>Method II</u> (mapping to a finite ~~domain~~ interval)



Let $u = e^{-x}$.

$$\int_0^\infty f(x) e^{-x}\,dx = \left| \begin{array}{l} u = e^{-x} \Rightarrow x = -\ln u \\ du = -\underbrace{e^{-x}}_{u}\,dx = -u\,dx \\ x = 0 \Rightarrow u = 1 \\ x = \infty \Rightarrow u = 0 \end{array} \right| =$$

$$= \int_1^0 f(-\ln u)(-du) = \int_0^1 \underbrace{f(-\ln u)}_{\text{new integrand}}\,du$$

Now we can use Trapezoid, Simpson or Gauss quadrature.

Method 3 (Gauss-Laguerre Quadrature)

Define a new inner product

$$\langle f, g \rangle = \int_0^\infty f(x) g(x) \underbrace{e^{-x}}_{\text{weight function}} \, dx$$

Check

1. $\langle f, f \rangle \geq 0$ and $\langle f, f \rangle = 0$ iff $f = 0$.
2. $\langle f, \alpha g + h \rangle = \alpha \langle f, g \rangle + \langle f, h \rangle$

The Laguerre polynomials are orthogonal with respect to this new inner product. They can be obtained using Gram-Schmidt orthogonalization process applied to $\{1, x, x^2, \dots\}$

$$L_0(x) = 1$$
$$L_1(x) = x - 1$$
$$L_2(x) = x^2 - 4x + 2$$
$$\dots$$

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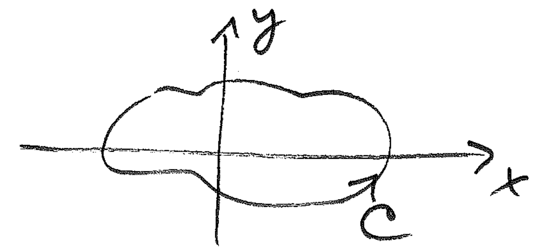
__Aside__

$$L_n(z) = \frac{1}{2\pi i} \oint_C \frac{e^{-zt(1-t)}}{(1-t)\,t^{n+1}}\, dt$$



Let $x_1, x_2, \dots, x_n$ be roots of polynomial $L_n(x)$. Note, $\deg L_n = n$ and roots $x_1, \dots, x_n \in (0, \infty)$.

$$c_j = \int_0^\infty \underbrace{f_j(x)}_{\text{Lagrange polynomials}} e^{-x}\, dx$$

Then

$$\boxed{\int_0^\infty f(x) e^{-x}\, dx \sim \sum_{j=1}^n c_j f(x_j)}$$

__Gauss-Laguerre quadrature__

where x_j, $j=1,\dots,n$ are roots of $L_n(x)$. This formula is exact for polynomials of degree $\le 2n-1$.

Ex The 2-point Gauss-Laguerre quadrature

$$L_2(x) = x^2 - 4x + 2 = x^2 - 4x + 4 - 2 = (x-2)^2 - 2$$

$$\Rightarrow x_1 = 2 - \sqrt{2}, \qquad x_2 = 2 + \sqrt{2}$$

$$\int_0^\infty f(x) e^{-x} dx \sim \underbrace{\frac{\sqrt{2}+1}{2\sqrt{2}}}_{c_1} f(2-\sqrt{2}) + \underbrace{\frac{\sqrt{2}-1}{2\sqrt{2}}}_{c_2} f(2+\sqrt{2}) \sim$$

$$\sim 0.854 \cdot f(0.586) + 0.146\, f(3.414)$$

Note

$$c_j = \int_0^\infty l_j(x) e^{-x} dx$$

or one can use the method of undetermined coefficients to find c_1 and c_2

$f(x) = 1$ $\qquad \int_0^\infty 1 \cdot e^{-x} dx = c_1 f(x_1) + c_2 f(x_2)$

$f(x) = x$ $\qquad \int_0^\infty x^2 e^{-x} dx = c_1 f(x_1) + c_2 f(x_2)$

Initial value problem (IVP) for ordinary differential equations (ODEs)

Find a function $y(t)$ that satisfies the equation $y' = f(y)$, $y = y(t)$, subject to initial condition $y(0) = y_0$.

In general,

$$\underbrace{\frac{dy}{dt}}_{y'} = f(t, y), \qquad y(t_0) = y_0$$

Ex

(a) $y' = y$, $y(0) = 1$

$\frac{dy}{dt} = y$: separable ODE

$y \neq 0$ $\quad \frac{dy}{y} = dt \quad \Rightarrow \quad \int \frac{dy}{y} = \int dt$

$\ln|y| = t + \tilde{c}$ $\quad | \exp$ $\qquad e^{\ln|y|} = e^{t+\tilde{c}} \Rightarrow |y| = e^t \cdot e^{\tilde{c}}$

$y = ce^t$

IC: $y(0) = 1 \Rightarrow \quad 1 = ce^0 \Rightarrow c = 1 \quad \Rightarrow \quad \boxed{y(t) = e^t}$

(b) $y' = y^2, \quad y(0) = 1$

$\Rightarrow y(t) = \dfrac{1}{1-t}$: blows up at $t = 1$

(c) $y' = y^{1/2}, \quad y(0) = 0$

$\dfrac{dy}{dt} = y^{1/2}$

$\dfrac{dy}{\sqrt{y}} = dt, \qquad y \neq 0$

$2\sqrt{y} = t + \tilde{c}$

$y(0) = 0 \Rightarrow \tilde{c} = 0$

$\Rightarrow 2\sqrt{y} = t \Rightarrow \sqrt{y} = \dfrac{t}{2}$

$y = \dfrac{t^2}{4}$

Check if $y \equiv 0$ is a solution

$y' = 0 \Rightarrow 0 = \sqrt{0}, \quad y(0) = 0$

$\Rightarrow y \equiv 0$ is ~~also~~ also a solution

$\therefore \quad y = \begin{cases} \frac{t^2}{4} \\ 0 \end{cases} \quad : \ 2 \text{ solutions}$

(d) $y' = \sin y, \quad y(0) = 1 \quad \Rightarrow \quad y(t) - ?$

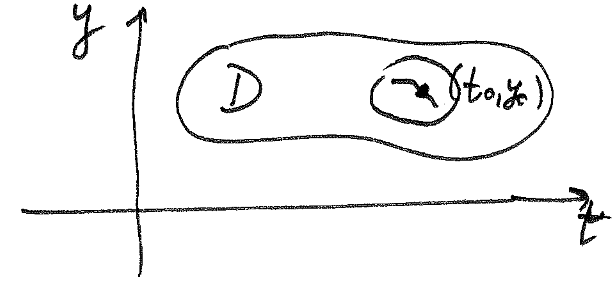
<u>**Def**</u>

An initial value problem is <u>well-posed</u> if the following properties are satisfied.

1. A solution <u>exists</u>
2. The solution is <u>unique</u>
3. The solution <u>depends continuously</u> on f and y_0

(small change in f and/or y_0 will result in small change in solution y).

<u>Thm</u> If $|f_y(y)|$ is bounded for $y \in D$, then the initial value problem $y' = f(y)$ subject to IC $y(0) = y_0$ is locally well-posed for all $y_0 \in D$.

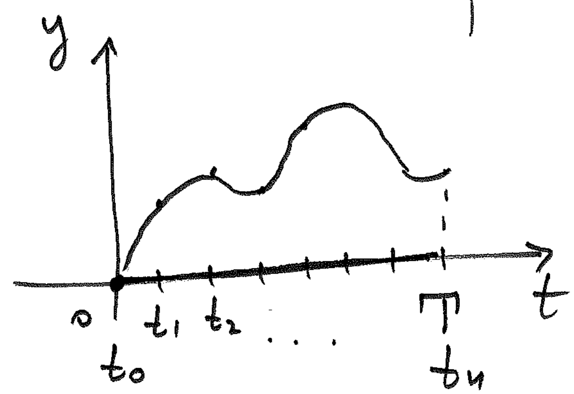


Euler's method

$h = \Delta t$ = step size

$t_n = nh, \quad t_0 = 0$

or $t_n = t_0 + nh$



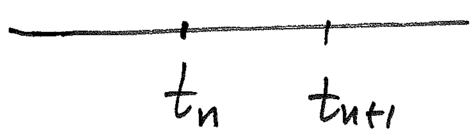
$t_0 = 0$

$u_n \approx y(t_n)$

approximation exact

$y' = f(y)$

$\frac{dy}{dt} = f(y)$



$$\frac{u_{n+1} - u_n}{h} = f(u_n) \ : \quad \text{forward difference formula}$$

$$\Rightarrow \boxed{u_{n+1} = u_n + h\, f(u_n), \qquad u_0 = y_0}$$