Lecture 36

Systems

$$y_1' = f_1(y_1, \dots, y_N) \qquad ' = \frac{d}{dt}$$

$$- - - -$$

$$y_N' = f_N(y_1, \dots, y_N)$$

Vector form

$$y' = f(y) \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \qquad f = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}$$

Special case

$$y_1' = a y_1 + b y_2$$

$$y_2' = c y_1 + d y_2$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$y' = Ay \qquad f(y) = Ay$$

$$f(u_n) = A \cdot u_n$$

$$u_{n+1} = u_n + h\, f(u_n)$$

$$u_{n+1} = u_n + h \cdot A u_n = (I + hA)\, u_n$$

Modified Euler's method

$$f(y) = Ay$$

$$k_1 = A u_n$$

$$k_2 = A(u_n + h \underbrace{k_1}_{= A u_n}) = A(u_n + hAu_n) = A(I + hA)u_n =$$

$$= (A + hA^2)\, u_n$$

$$u_{n+1} = u_n + \frac{h}{2}(k_1 + k_2) = u_n + \frac{h}{2}\left(Au_n + (A + hA^2)u_n\right) =$$

$$= \left(I + \frac{h}{2}A + \frac{h}{2}A + \frac{h^2}{2}A^2\right) u_n$$

$$\boxed{u_{n+1} = \left(I + hA + \frac{h^2}{2}A^2\right) u_n}$$

Exact solution

$$y' = Ay, \quad y(0) = y_0 \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Assume that matrix A can be diagonalized with eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ and eigenvectors p_1, p_2.

$$A p_1 = \lambda_1 p_1, \qquad A p_2 = \lambda_2 p_2, \qquad p_1 \neq \vec{0}, \; p_2 \neq \vec{0}$$

If $\lambda_1 \neq \lambda_2$, then e'vectors p_1 and p_2 are linearly independent and can form a basis in the solution space.

Let $y(t)$ be a solution of $y' = Ay$, then

$$y(t) = \alpha_1(t) p_1 + \alpha_2(t) p_2$$

at $t=0$:

$$y(0) = \alpha_1(0) p_1 + \alpha_2(0) p_2 = y_0$$

This determines $\alpha_1(0)$ and $\alpha_2(0)$.

$$y' = Ay \Rightarrow \alpha_1' p_1 + \alpha_2' p_2 = A(\alpha_1 p_1 + \alpha_2 p_2)$$

$$\alpha_1' p_1 + \alpha_2' p_2 = \alpha_1 \underbrace{A p_1}_{\lambda_1 p_1} + \alpha_2 \underbrace{A p_2}_{\lambda_2 p_2}$$

$$\alpha_1' p_1 + \alpha_2' p_2 = \lambda_1 \alpha_1 p_1 + \lambda_2 \alpha_2 p_2$$

p_1, p_2 are linearly independent $\Rightarrow$

$\alpha_1' = \lambda_1 \alpha_1, \quad \alpha_2' = \lambda_2 \alpha_2$: scalar DEs for α_1, α_2

differential eq$^{\text{ns}}$

$$\Rightarrow \boxed{y(t) = \underbrace{\alpha_1(0) e^{\lambda_1 t}}_{\alpha_1(t)} p_1 + \underbrace{\alpha_2(0) e^{\lambda_2 t}}_{\alpha_2(t)} p_2}$$

<u>exact solution</u>

Numerical solution

Ex: Euler's method

$$u_{n+1} = u_n + h\,\underbrace{f(u_n)}_{A u_n} = u_n + hAu_n = (I + hA)u_n$$

$$u_0 = y_0 = \alpha_1(0)\,p_1 + \alpha_2(0)\,p_2$$

$$u_1 = (I + hA)u_0 = (I + hA)\left(\alpha_1(0)\,p_1 + \alpha_2(0)\,p_2\right) =$$

$$= \alpha_1(0)\,p_1 + \alpha_2(0)\,p_2 + h\alpha_1(0)\underbrace{Ap_1}_{\lambda_1 p_1} + h\alpha_2(0)\underbrace{Ap_2}_{\lambda_2 p_2} =$$

$$= \alpha_1(0)(1 + h\lambda_1)\,p_1 + \alpha_2(0)(1 + h\lambda_2)\,p_2$$

$$u_2 = \alpha_1(0)(1 + h\lambda_1)^2 p_1 + \alpha_2(0)(1 + h\lambda_2)^2 p_2$$

$$\boxed{u_n = \alpha_1(0)(1 + h\lambda_1)^n p_1 + \alpha_2(0)(1 + h\lambda_2)^n p_2}$$ numerical solution

$$\boxed{y(t) = \alpha_1(0)\, e^{\lambda_1 t}\, p_1 + \alpha_2(0)\, e^{\lambda_2 t}\, p_2} : \text{ exact solution}$$

Note

The exact solution is bounded for all $t>0$ and all y_0

$\Leftrightarrow$ $Re(\lambda_1) \leq 0$ and $Re(\lambda_2) \leq 0$

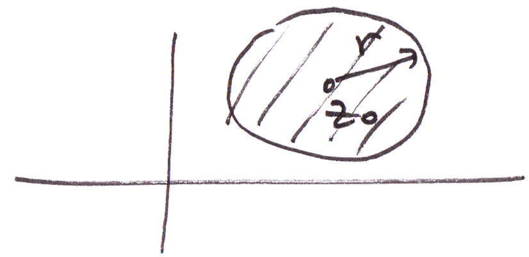
Numerical solution is bounded for all $n>0$ and all y_0

$\Leftrightarrow$ $|1+h\lambda_1| \leq 1$ and $|1+h\lambda_2| \leq 1$

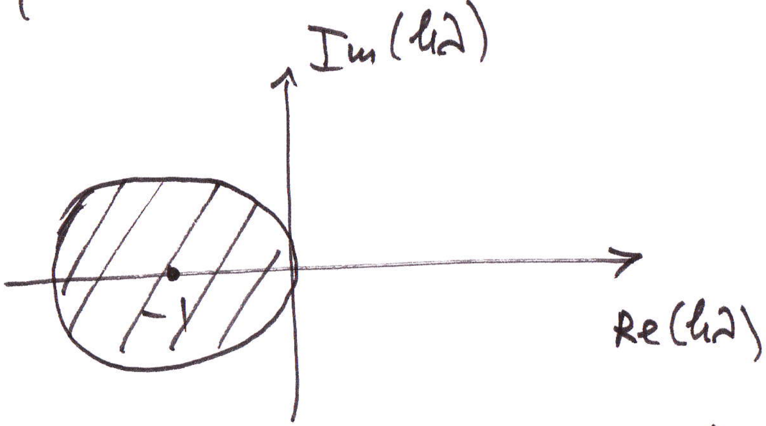
The region in $h\lambda$-plane which satisfies these conditions is called the region of absolute stability for Euler's method.

$$|1+h\lambda| \le 1 \qquad |1+z| \le 1 \quad \text{or} \quad |z+1| \le 1$$

$$|h\lambda+1| \le 1 \qquad |z-z_0| \le r$$

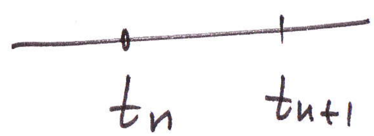


$$|h\lambda-(-1)| \le 1$$



Backward Euler method

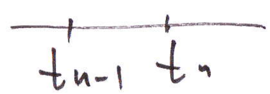
$$y' = f(y)$$



$$\frac{u_{n+1}-u_n}{h} = f(u_{n+1})$$

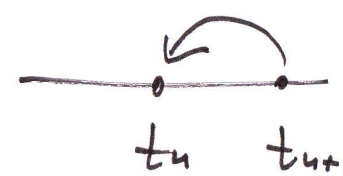
$$\boxed{u_{n+1} = u_n + h\, f(u_{n+1})}$$

$$\frac{u_n-u_{n-1}}{h} = f(u_n)$$



Forward Euler method

$$\frac{u_{n+1}-u_n}{h} = f(u_n)$$



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This method is <u>implicit</u> (Forward Euler is <u>explicit</u>)

<u>Claim</u> Backward Euler's method is 1st order accurate.

<u>Pf</u>

$y' = f(y)$

$$u_{n+1} = u_n + h f(u_{n+1})$$

$$y_{n+1} = y_n + h f(y_{n+1}) + r_n \quad : \quad \text{for exact solution } y(t)$$

r_n local truncation error

$$y_{n+1} = y(t_{n+1}) = y(t_n + h) \overset{\text{Taylor}}{=} y(t_n) + \underbrace{y'(t_n)}_{f(t_n)} \cdot h + O(h^2)$$

$$\Rightarrow y_n + h f(y_{n+1}) + r_n = \underbrace{y(t_n)}_{y_n} + \underbrace{y'(t_n)}_{f(t_n)} \cdot h + O(h^2)$$

$$f(y_{n+1}) = y'_{n+1} = y'(t_{n+1}) = y'(t_n + h) \overset{\text{Taylor}}{=} y'(t_n) + y''(t_n)\cdot h + O(h^2)$$

$$y_n + h\left(y'(t_n) + y''(t_n)\cdot h + O(h^2)\right) + r_n = y_n + y'(t_n)\cdot h + O(h^2)$$

$$\Rightarrow \underbrace{h^2 y''(t_n) + h\cdot O(h^2)}_{O(h^2)} + r_n = O(h^2) \Rightarrow \boxed{r_n = O(h^2)}$$

Since the local truncation error is 2^{nd} order accurate $\Rightarrow$ the global error is 1^{st} order accurate, ie. Backward Euler's method is 1^{st} order accurate.