University of Idaho

Lecture 37

HW problem #2: see lecture #29

Idea is similar to derivation of local truncation error in trapezoid rule, but one needs to use 3 points $x_0 = a$, $x_1 = a + h + \varepsilon$ and $x_2 = b$, $h = \frac{b-a}{2}$, and interpolate $f(x)$ with 2nd degree interpolating polynomial

$$f(x) = p_2(x) + \frac{f'''(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2)$$

exact — approx. — error

$$\int_a^b f(x)\,dx = \int_a^b p_2(x)\,dx + \int_a^b \frac{f'''(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2)\,dx$$

exact — quadrature — error in quadrature

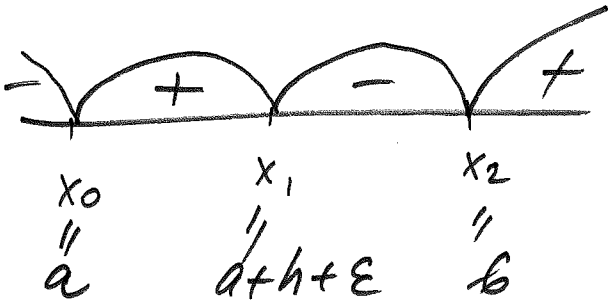
Recall, generalized MVT:

$$\int_a^b f(x)g(x)\,dx = f(\hat{\xi}) \int_a^b g(x)\,dx, \qquad g(x) \geq 0$$

If we use $p_1(x)$, the 1st degree interpolating polynomial, as in the derivation of trapezoid rule, $(x-x_0)(x-x_1)$ does not change sign on $[a, a+h]$.

($x_0 = a$, $x_1 = a+h$)

In our case, $(x-x_0)(x-x_1)(x-x_2) = w_2(x)$



$w_2(x)$ changes its sign inside $[a, b]$.

Hence, we cannot use generalized MVT directly.

Remedy: apply integration by parts to

$$\int_a^b \underbrace{\frac{f'''(\xi(x))}{3!}}_{dv} \underbrace{(x-x_0)(x-x_1)(x-x_2)\,dx}_{u}$$

$u = (x-x_0)(x-x_1)(x-x_2)$: cubic polynomial

$du = \left(\begin{array}{c}\text{polynomial of degree 2 with roots that are not inside } [a, b]\end{array}\right) dx$

$$dv = \frac{f'''(\xi(x))}{3!}\,dx$$

$$v = \frac{f''(\xi(x))}{3!}\,\frac{1}{\xi'}$$

Recall integration by parts formula:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Ex $$\int_0^\pi x\sin x\,dx = \left| \begin{array}{ll} u = x & dv = \sin x\,dx \\ du = dx & v = -\cos x \end{array} \right| = -x\cos x\Big|_0^\pi + \int_0^\pi \cos x\,dx =$$

$$= \pi + \sin x\Big|_0^\pi = \pi$$

Then apply generalized MVT.

$$\int_a^b p_2(x)\,dx = \text{quadrature}$$

Use some symbolic software like Maple, Mathematica (there is free online version of Mathematica) or Matlab w/ symbolic variables (Matlab uses Maple commands). The installed version of Matlab has symbolic tool box.

Back to Backward Euler's method

Absolute stability

$y' = \lambda y$ $\qquad f(y) = \lambda y$

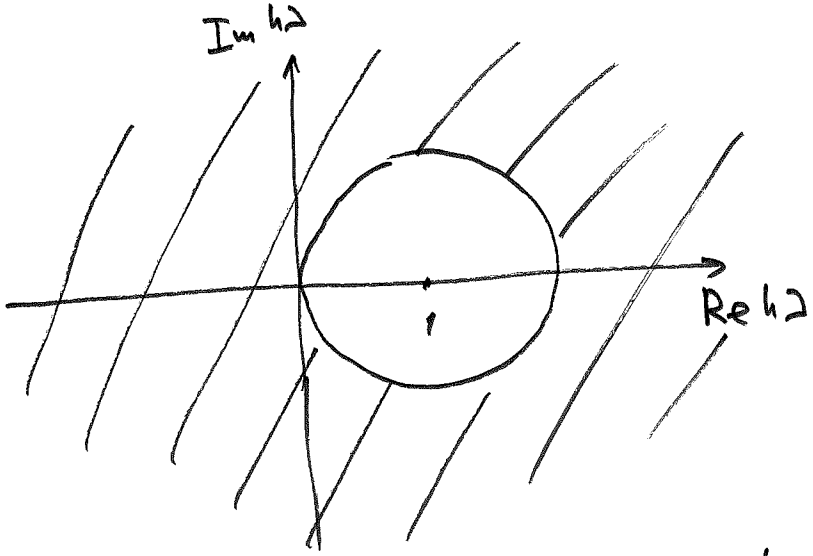
Backward Euler: $u_{n+1} = u_n + h f(u_{n+1}) = u_n + h\lambda u_{n+1}$

($f(u_{n+1}) = \lambda u_{n+1}$)

$$\Rightarrow (1 - h\lambda) u_{n+1} = u_n \Rightarrow u_{n+1} = \underbrace{\frac{1}{1-h\lambda}}_{\text{amplification factor for Backward Euler}} u_n$$

$$\Rightarrow u_n = \left(\frac{1}{1-h\lambda}\right)^n u_0$$

u_n is bounded for all $n \Rightarrow \left|\frac{1}{1-h\lambda}\right| \leq 1$ or $|h\lambda - 1| \geq 1$



Since $\mathrm{Re}\,\lambda < 0$, we have no restriction on h

Method is called <u>A-stable</u> if the region of absolute stability includes the entire half-plane.

<u>Def</u> a system of ODEs, $y' = Ay$, is called a <u>stiff system</u> if A has negative e'values with greatly different magnitudes.

<u>Ex</u> $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Assume $\lambda_1 << \lambda_2 < 0$

$$y(t) = \alpha_1(0)\, e^{\lambda_1 t} p_1 + \alpha_2(0)\, e^{\lambda_2 t} p_2 : \text{ exact}$$

$$u_n = \alpha_1(0)\,(1+h\lambda_1)^n p_1 + \alpha_2(0)(1+h\lambda_2)^n p_2 : \text{ Euler's method}$$

$$u_n = \alpha_1(0)\left(\frac{1}{1-h\lambda_1}\right)^n p_1 + \alpha_2(0)\left(\frac{1}{1-h\lambda_2}\right)^n p_2 : \text{ backward Euler's method}$$

We require that both $h\lambda_1$ and $h\lambda_2$ be in the region of absolute stability. For Euler's method, we impose

$$\boxed{h \le \frac{-1}{\lambda_1}}, \quad h \le -\frac{1}{\lambda_2} \qquad \lambda_1 \ll \lambda_2 < 0$$

small

For backward Euler, there is no restriction on h.

→ we can use larger h and we choose h to have some accuracy.

Note

Implicit schemes are implemented as predictor-corrector methods.

Ex $u_{n+1} = u_n + h f(u_{n+1})$

$\Rightarrow$ $u_{n+1}^{(1)} = u_n + h f(u_n)$: predictor

$u_{n+1}^{(k+1)} = u_n + h f(u_{n+1}^{(k)})$: corrector

```
> x0:=a;
```

$$x0 := a \tag{1}$$

```
> x1:=a+h+epsilon;
```

$$x1 := a + h + \varepsilon \tag{2}$$

```
> x2:=a+2*h;
```

$$x2 := a + 2\,h \tag{3}$$

```
> simplify(int((x-x0)*(x-x1),x=x0..x2));
```

$$\frac{2}{3}\,h^3 - 2\,\varepsilon\,h^2 \tag{4}$$

```
> f0:=f(x0);
```

$$f0 := f(a) \tag{5}$$

```
> f01:=(f(x1)-f(x0))/(x1-x0);
```

$$f01 := \frac{f(a+h+\varepsilon) - f(a)}{h+\varepsilon} \tag{6}$$

```
> f12:=(f(x2)-f(x1))/(x2-x1);
```

$$f12 := \frac{f(a+2\,h) - f(a+h+\varepsilon)}{h-\varepsilon} \tag{7}$$

```
> f012:=simplify((f12-f01)/(x2-x0));
```

$$f012 := -\frac{1}{2}\,\frac{f(a+2\,h)\,h + f(a+2\,h)\,\varepsilon - 2f(a+h+\varepsilon)\,h + f(a)\,h - f(a)\,\varepsilon}{h\left(-h^2+\varepsilon^2\right)} \tag{8}$$

```
> p2:=f0+f01*(x-x0)+f012*(x-x0)*(x-x1);
```

$$p2 := f(a) + \frac{\left(f(a+h+\varepsilon) - f(a)\right)(x-a)}{h+\varepsilon} - \frac{1}{2}\,\frac{1}{h\left(-h^2+\varepsilon^2\right)}\left(\left(f(a+2\,h)\,h + f(a+2\,h)\,\varepsilon - 2f(a+h+\varepsilon)\,h + f(a)\,h - f(a)\,\varepsilon\right)(x-a)\,(x-a-h-\varepsilon)\right) \tag{9}$$

```
> p2int:=simplify(int(p2(x),x=x0..x2));
```

$$p2int := \frac{1}{2}\int_a^{a+2\,h} \frac{1}{h(x)\left(h(x)^2 - \varepsilon(x)^2\right)}\Big(2f(a+2\,h)(x)\,h(x)\,a(x)\,\varepsilon(x) + f(a)(x)\,x(x)\,\varepsilon(x)^2 - 2f(a+2\,h)(x)\,h(x)\,x(x)\,a(x) - 3f(a)(x)\,x(x)\,h(x)^2 - 4f(a+h+\varepsilon)(x)\,a(x)\,h(x)^2 + 4f(a+h+\varepsilon)(x)\,x(x)\,h(x)^2 + 3f(a)(x)\,a(x)\,h(x)^2 - f(a)(x)\,a(x)\,\varepsilon(x)^2 + f(a)(x)\,h(x)\,a(x)^2 - f(a + 2\,h)(x)\,\varepsilon(x)^2\,x(x) + f(a+2\,h)(x)\,\varepsilon(x)^2\,a(x) - 2f(a+h + \varepsilon)(x)\,h(x)\,x(x)^2 - 2f(a+h+\varepsilon)(x)\,h(x)\,a(x)^2 + f(a)(x)\,h(x)\,x(x)^2 - 2\,\varepsilon(x)^2 f(a)(x)\,h(x) + f(a+2\,h)(x)\,\varepsilon(x)\,a(x)^2 - f(a)(x)\,\varepsilon(x)\,x(x)^2 - f(a)(x)\,\varepsilon(x)\,a(x)^2 + f(a+2\,h)(x)\,h(x)\,x(x)^2 - f(a+2\,h)(x)\,h(x)^2\,x(x) \tag{10}$$

$$+f(a+2h)(x)\,h(x)\,a(x)^2+f(a+2h)(x)\,h(x)^2\,a(x)+f(a$$
$$+2h)(x)\,\varepsilon(x)\,x(x)^2+2f(a)(x)\,h(x)^3-2f(a+2h)(x)\,\varepsilon(x)\,x(x)\,a(x)$$
$$+2f(a)(x)\,h(x)\,x(x)\,\varepsilon(x)-2f(a)(x)\,h(x)\,a(x)\,\varepsilon(x)$$
$$+2f(a)(x)\,\varepsilon(x)\,x(x)\,a(x)-2f(a)(x)\,h(x)\,x(x)\,a(x)+4f(a+h$$
$$+\varepsilon)(x)\,h(x)\,x(x)\,a(x)-2f(a+2h)(x)\,h(x)\,x(x)\,\varepsilon(x))\,dx$$

```
> p2intd:=collect(simplify(f(x0)*(x2-x0)+f01*(x2-x0)^2+f012*(2/3*
  h^3-2*epsilon*h^2)),f);
```

$$p2intd := \frac{1}{3}\,\frac{\left(3\varepsilon^2-8h\varepsilon+5h^2\right)hf(a)}{-h^2+\varepsilon^2}+\frac{1}{3}\,\frac{\left(6h\varepsilon-10h^2\right)hf(a+h+\varepsilon)}{-h^2+\varepsilon^2} \tag{11}$$
$$+\frac{1}{3}\,\frac{\left(3\varepsilon^2+2h\varepsilon-h^2\right)hf(a+2h)}{-h^2+\varepsilon^2}$$

```
> eval(p2intd,epsilon=0);
```

$$-\frac{5}{3}f(a)\,h+\frac{10}{3}\,hf(a+h)+\frac{1}{3}f(a+2h)\,h \tag{12}$$

```
> v:=collect(int((x-x0)*(x-x1)*(x-x2),x),x);
```

$$v := \frac{1}{4}x^4+\left(-a-h-\frac{1}{3}\varepsilon\right)x^3+\left(-\frac{1}{2}a\left(-a-h-\varepsilon\right)+\frac{1}{2}\left(-2a-h-\varepsilon\right)(-a\right. \tag{13}$$
$$\left.-2h)\right)x^2-a\left(-a-h-\varepsilon\right)(-a-2h)\,x$$

```
> eval(v,x=a);
```

$$\frac{1}{4}a^4+\left(-a-h-\frac{1}{3}\varepsilon\right)a^3+\left(-\frac{1}{2}a\left(-a-h-\varepsilon\right)+\frac{1}{2}\left(-2a-h-\varepsilon\right)(-a\right. \tag{14}$$
$$\left.-2h)\right)a^2-a^2\left(-a-h-\varepsilon\right)(-a-2h)$$

```
> f2x:=(f(x2)-f(x))/(x2-x);
```

$$f2x := \frac{f(a+2h)-f(x)}{a+2h-x} \tag{15}$$

```
> f12x:=(f2x-f12)/(x-x1);
```

$$f12x := \frac{\dfrac{f(a+2h)-f(x)}{a+2h-x}-\dfrac{f(a+2h)-f(a+h+\varepsilon)}{h-\varepsilon}}{x-a-h-\varepsilon} \tag{16}$$

```
> f012x:=(f12x-f012)/(x-x0);
```

$$f012x := \frac{1}{x-a}\left(\frac{\dfrac{f(a+2h)-f(x)}{a+2h-x}-\dfrac{f(a+2h)-f(a+h+\varepsilon)}{h-\varepsilon}}{x-a-h-\varepsilon}\right. \tag{17}$$

$$+\frac{1}{2}\,\frac{f(a+2h)\,h+f(a+2h)\,\varepsilon-2f(a+h+\varepsilon)\,h+f(a)\,h-f(a)\,\varepsilon}{h\left(-h^2+\varepsilon^2\right)}\Bigg)$$

```
> eval(f012x,x=x0);
```

```
Error, numeric exception: division by zero
```

```
> simplify(int((x-x0)*(x-x1)*(x-x2),x=x0..x2));
```

$$\frac{4}{3}\,\varepsilon\,h^3 \tag{18}$$

```
> simplify(int(collect(int((tau-x0)*(tau-x1)*(tau-x2),tau=x0..x),
  x),x=x0..x2));
```

$$\frac{4}{15}\,h^5+\frac{4}{3}\,\varepsilon\,h^4 \tag{19}$$

```
> g:=collect(int((tau-x0)*(tau-x1)*(tau-x2),tau=x0..x),x);
```

$$g:=\frac{1}{4}\,x^4+\left(-a-h-\frac{1}{3}\,\varepsilon\right)x^3+\Bigg(-\frac{1}{2}\,a\left(-a-h-\varepsilon\right)+\frac{1}{2}\left(-2\,a-h-\varepsilon\right)(-a-2\,h)\Bigg)x^2-a\left(-a-h-\varepsilon\right)(-a-2\,h)\,x-\frac{1}{2}\left(-a\left(-a-h-\varepsilon\right)+\left(-2\,a-h-\varepsilon\right)(-a-2\,h)\right)a^2-\frac{1}{4}\,a^4-\frac{1}{3}\left(-3\,a-3\,h-\varepsilon\right)a^3+a^2\left(-a-h-\varepsilon\right)(-a-2\,h) \tag{20}$$

```
> collect(limit(f012x,x=x2),h);
```

$$-\frac{1}{4}\,\frac{1}{h^2\left(-h^2+\varepsilon^2\right)\left(-h+\varepsilon\right)}\Big(-2\,\mathrm{D}(f)\,(a+2h)\,h^3+\left(3f(a+2h)-4f(a+h+\varepsilon)+f(a)\right)h^2+\left(2\,\mathrm{D}(f)\,(a+2h)\,\varepsilon^2+2f(a+2h)\,\varepsilon-2f(a)\,\varepsilon\right)h-f(a+2h)\,\varepsilon^2+f(a)\,\varepsilon^2\Big) \tag{21}$$

```
> simplify(int((tau-x0)*(tau-x1)*(tau-x2),tau=x0..x2));
```

$$\frac{4}{3}\,\varepsilon\,h^3 \tag{22}$$

```
>
```