Lecture 3

University of Idaho

We showed last time

$$f'(x) = \underbrace{\frac{f(x+h) - f(x)}{h}}_{D_+ f} + \underbrace{O(h)}_{\text{error} \approx C\cdot h}$$

$p=1$

Error is a linear function of h.

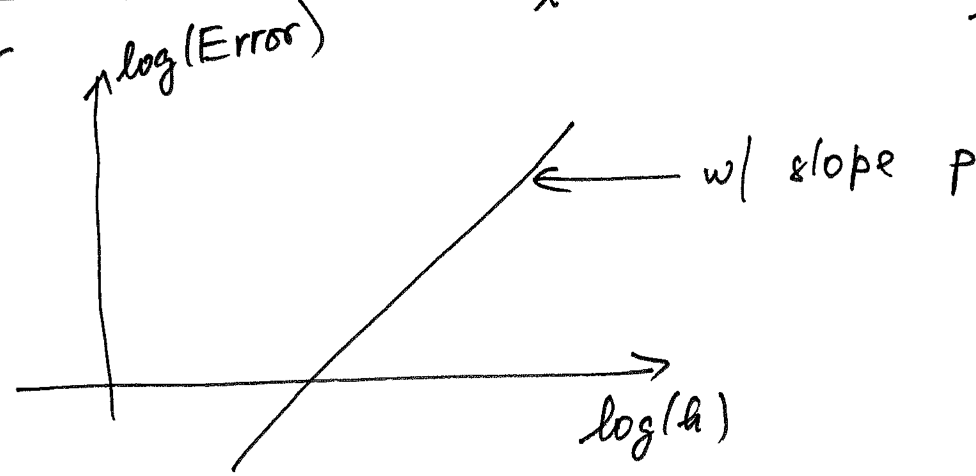
In general,

Error $\sim C\cdot h^p$ $\qquad$ p: order of convergence

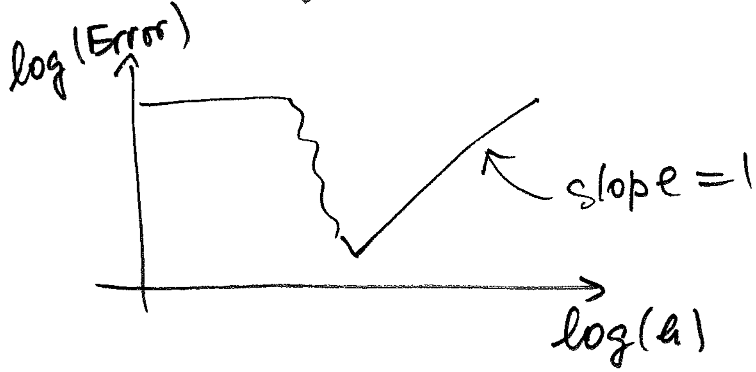
$$\underbrace{\log(\text{Error})}_{y} \sim \log C + p\,\underbrace{\log(h)}_{x}$$

$\log(a\cdot b) = \log a + \log b$

$\log a^n = n \log a$



University of Idaho



that is why we plot $\log(\text{Error})$ vs. $\log(h)$ and not just Error vs. h.

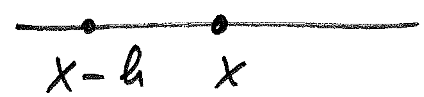
Q why does the error increase for small h?

1. The computed value has two sources of error: truncation error is due to replacing $f'(x)$ by finite difference approximation $D_+ f(x)$, and roundoff error is due to finite precision computer arithmetic.
2. The truncation error is $O(h)$ and the roundoff error is $O(\varepsilon/h)$ where $\varepsilon \approx 10^{-15}$ in Matlab.

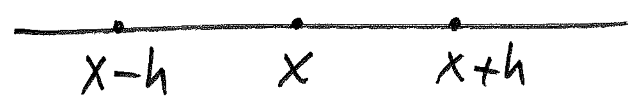
3. The total error is $O(h) + O(\varepsilon/h)$.
Hence, for large h, the truncation error dominates the roundoff error, but for small h, the roundoff error dominates.

<u>Note</u>

$$D_- f(x) = \frac{f(x) - f(x-h)}{h}$$: <u>backward difference</u> or <u>left-side difference</u>



$$D_0 f(x) = \frac{f(x+h) - f(x-h)}{2h}$$: <u>central difference</u>



One can show

$$f'(x) = D_- f + O(h); \qquad f'(x) = D_0 f + O(h^2)$$

Ch 2. Root finding

Bisection Method

Def Given $f(x)$, a number p satisfying $f(p)=0$ is called a root of $f(x)$.

Ex $f(x) = x^2 - 3x + 2 \Rightarrow p = 1, 2$

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

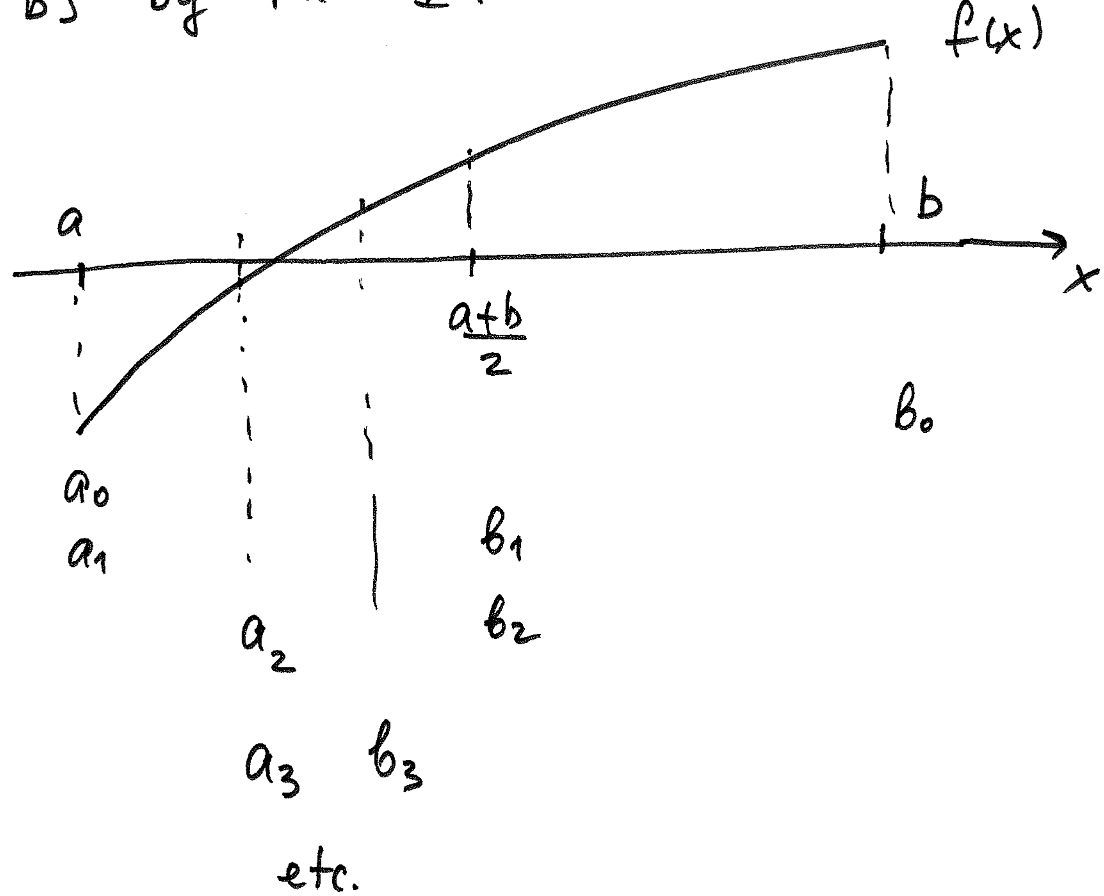
x_1, x_2: roots

Vietta's Thm: $x_1 \cdot x_2 = \frac{c}{a}$, $x_1 + x_2 = -\frac{b}{a}$

Ex $f(x) = x^2 - 3 \Rightarrow p = \pm\sqrt{3}$

Q How do we find roots of a general function?

Idea: Find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite sign. Then $f(x)$ has a root in $[a, b]$ by the Intermediate Value Thm.



Consider the midpoint $\frac{a+b}{2}$. The root is contained in either left subinterval $[a, \frac{a+b}{2}]$ or right subinterval $[\frac{a+b}{2}, b]$. Determine on which subinterval f assumes values of opposite sign by computing $f(\frac{a+b}{2})$ and comparing w/ $f(a)$ and $f(b)$. Then repeat.

<u>Bisection method</u> (assume $f(a) \cdot f(b) < 0$)

1. $n = 0$, $a_0 = a$, $b_0 = b$
2. $x_n = \frac{a_n + b_n}{2}$: current estimate of the root
3. if $f(x_n) \cdot f(a_n) < 0$, then $a_{n+1} = a_n$, $b_{n+1} = x_n$
4. else $a_{n+1} = x_n$, $b_{n+1} = b_n$
5. set $n = n+1$ and go to line 2

University of Idaho

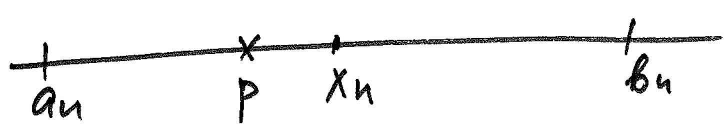
Ex $f(x) = x^2 - 3$, $f(1) = -2 < 0$, $f(2) = 1 > 0$

$\Rightarrow$ there is a root p in $[1, 2]$

$p = \sqrt{3} = 1.73205$

n	a_n	b_n	x_n	$f(x_n)$	$\lvert p - x_n \rvert$
0	1	2	1.5	−0.75	0.2321
1	1.5	2	1.75	0.0625	0.0179
2	1.5	1.75	1.625	−0.3594	0.1071
3	1.625	1.75	1.6875	−0.1523	0.0446
4	1.6875	1.75	1.71875	−0.0459	0.0133

Error bound for the bisection method



$$|p - x_n| \le |b_n - a_n| = \frac{1}{2}|b_{n-1} - a_{n-1}| = \dots = \left(\frac{1}{2}\right)^n |b_0 - a_0|$$

$$\Rightarrow \boxed{|p - x_n| \le \left(\frac{1}{2}\right)^n |b - a|}$$

University of Idaho

<u>Ex</u> How many steps are needed to ensure that the error is less than 10^{-3} ?

$a=1,\ b=2$

$$|\text{error}| \le \left(\tfrac{1}{2}\right)^n \underbrace{\overset{2-1}{|b-a|}}_{=1} \le 10^{-3}$$

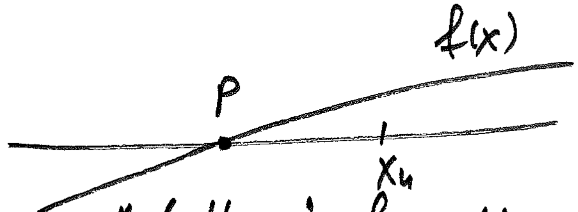
$$\left(\tfrac{1}{2}\right)^n \le 10^{-3}$$

$$\frac{1}{2^n} \le \frac{1}{10^3} \Rightarrow 2^n \ge 10^3 = 1000 \qquad 2^{10} = 1024$$

$$\Rightarrow \boxed{n \ge 10}$$

<u>Stopping criteria</u>: there are 3 options.

$|b_n - a_n| < \varepsilon$, $\quad |f(x_n)| < \varepsilon \quad\quad n = n_{max}$



"shallow" function