Lecture 42

Linear shooting method (Cont'd)

Recast as a system of 1^{st} order IVPs:

Let $u_1 = v$, $u_2 = v'$, $u_3 = w$, $u_4 = w'$

$$\Rightarrow \boxed{\begin{array}{ll} \left.\begin{array}{l} u_1' = u_2 \\ u_2' = p(x)\,u_2 + q(x)\,u_1 + r(x) \\ u_3' = u_4 \\ u_4' = p(x)\,u_4 + q(x)\,u_3 \end{array}\right\} & a \le x \le b \\ u_1(a) = \alpha, \quad u_2(a) = 0, \quad u_3(a) = 0, \quad u_4(a) = 1 & \end{array}}$$

$$\Rightarrow \boxed{u(x) = u_1(x) + \frac{\beta - u_1(b)}{u_3(b)}\, u_3(x)}$$

Note We can solve the above IVP using, for example $O(h^4)$ Runge-Kutta method for systems.

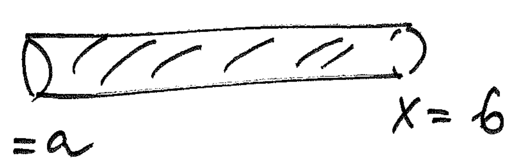
Summary

We could use 2 methods to solve linear heat equation

- linear shooting method: turn BVP into 2 IVSs for $v(x)$ and $w(x)$ $\Rightarrow u(x) = v(x) + C\,w(x)$
- linear finite difference: replace derivatives by finite differences, BVP $\Rightarrow$ linear algebraic system (sparse system – lots of zeros). Can solve by Gaussian elimination or iterative methods.

Nonlinear shooting method

Application: heat flow in a thin rod



x: position along rod

$u(x)$: steady-state temperature

$f(x,u,u')$: external heat source

The steady-state heat eqⁿ:

$$\left.\begin{aligned} u''(x) &= -f(x,u,u') \\ u(a) &= \alpha \\ u(b) &= \beta \end{aligned}\right\} \text{BVP}$$

<u>Thm</u>

Let $D = \{(x,u,u'): \quad a \le x \le b, \ -\infty < u < \infty, \ -\infty < u' < \infty\}$

If

1. $f, \dfrac{\partial f}{\partial u}, \dfrac{\partial f}{\partial u'}$ are continuous on D
2. $\dfrac{\partial f}{\partial u} < 0$ on D
3. $\left|\dfrac{\partial f}{\partial u'}\right| \le M$ on D

Then the above BVP has a unique solution.

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Idea: convert BVP to IVP.

IVP: $\left.\begin{aligned} u''(x) &= -f(x, u, u') \\ u(a) &= \alpha \\ u'(a) &= s \end{aligned}\right\}$ s is unknown

<u>Goal</u>: Find s such that $u(b) = \beta$. This is called a shooting method because we are aiming at u (by choosing s) so that we hit a target ($u = \beta$ at $x = b$)

<u>Choosing s</u>

The solution depends on x and s: $u = u(x, s)$

We want $u(b, s) = \beta \Rightarrow \overbrace{u(b, s) - \beta}^{F(s)} = 0$

This is a root finding problem.

$f(x) = 0$
Newton's method
$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Newton's method:

1. Start with initial guess s_0
2. $s_{k+1} = s_k - \left[\dfrac{u(b, s_k) - \beta}{\frac{\partial u}{\partial s}(b, s_k)} \right]$ for $k = 0, 1, 2, \ldots$

$u(b, s_k)$: solve IVP with $u'(a) = s_k$ and evaluate solution at $x = b$

Question : how do we compute $\frac{\partial u}{\partial s}(b, s_k)$?

Computing $\frac{\partial u}{\partial s}$

IVP:
$$u''(x,s) = -f(x, u(x,s), u'(x,s))$$
$$u(a,s) = \alpha$$
$$u'(a,s) = s$$

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Take partial derivative w/ s:

$$\frac{\partial u''}{\partial s} = -\frac{\partial f}{\partial s} = -\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial s} - \frac{\partial f}{\partial u'}\cdot\frac{\partial u'}{\partial s} \qquad (*)$$

$$u(a,s) = \alpha \Rightarrow \frac{\partial u}{\partial s}(a,s) = 0$$

$$u'(a,s) = s \Rightarrow \frac{\partial u'}{\partial s}(a,s) = 1$$

Let $z(x,s) = \frac{\partial u}{\partial s}(x,s)$

$$z'(x,s) = \frac{\partial}{\partial x}\frac{\partial u}{\partial s} = \frac{\partial}{\partial s}\frac{\partial u}{\partial x} = \frac{\partial u'}{\partial s} \qquad (**)$$

$$z''(x,s) = \frac{\partial}{\partial x}\frac{\partial u'}{\partial s} = \frac{\partial}{\partial s}\frac{\partial^2 u}{\partial x^2} = \frac{\partial u''}{\partial s}$$

$\therefore$ $z(x,s)$ must satisfy

IVP: $$z''(x,s) \underset{(**)}{\overset{(*)}{=}} -\frac{\partial f}{\partial u}(x,u,u')\cdot z(x,s) - \frac{\partial f}{\partial u'}(x,u,u')\, z'(x,s)$$

$$z(a,s) = 0, \quad z'(a,s) = 1$$

Converting to a 1st order system:

Let $u_1 = u, \quad u_2 = u', \quad u_3 = z, \quad u_4 = z'$

$$\Rightarrow \boxed{\begin{aligned} u_1' &= u_2 \\ u_2' &= -f(x, u_1, u_2) \\ u_3' &= u_4 \\ u_4' &= -\frac{\partial f}{\partial u}(x, u_1, u_2)\cdot u_3 - \frac{\partial f}{\partial u'}(x, u_1, u_2)\cdot u_4 \\ & u_1(a) = \alpha, \quad u_2(a) = s_k, \quad u_3(a) = 0, \quad u_4(a) = 1 \end{aligned}} \qquad (1)$$

General procedure

1. Choose s_0, set $k = 0$.
2. Solve IVP (1) with $u_2(a) = s_k$
3. Compute $s_{k+1} = s_k - \left[\dfrac{u_1(b, s_k) - \beta}{u_3(b, s_k)}\right]$

   $k = k+1$

4. Repeat step 2 and 3 until

$$|s_k - s_{k-1}| < tol$$

Notes

1. How quickly we converge depends on initial guess s_0.
2. Newton's method will converge quadratically because $z(b, s) = u_3(b, s_k) \neq 0$. Can show using uniqueness argument.
3. At each iteration, must use a numerical method such as $O(h^4)$ Runge-Kutta to solve the system of four 1st order DEs (IVP).