Lecture 44

Least squares w/ exponentials (Cont'd)

1) assume $y = \beta e^{\alpha x}$

$$\ln y = \ln\beta + \underbrace{\ln e^{\alpha x}}_{\alpha x} \Rightarrow \ln y = \ln\beta + \alpha x$$

$\ln y$ is a linear function of x with slope $= \alpha$ and free coefficient $\ln\beta$

$$A\vec{z} = \vec{b} \qquad \vec{z} = \begin{array}{c} \begin{pmatrix} \ln\beta \\ \alpha \end{pmatrix} \end{array}\begin{array}{l} a_0 \\ a_1 \end{array} = \Big(\qquad A = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_m \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} \ln y_1 \\ \ln y_2 \\ \vdots \\ \ln y_m \end{pmatrix}$$

$\therefore$ Taking $\ln$ of both sides of $y = \beta e^{\alpha x}$ reduced the exponential into a linear least squares problem

Solve: $A^T A \vec{z} = A^T \vec{b}$ for $\vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \ln\beta \\ \alpha \end{pmatrix}$

$$\Rightarrow \boxed{\beta = e^{z_1}, \quad \alpha = z_2}$$

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2) Assume $y = \beta x^{\alpha}$

$\ln y = \ln\beta + \ln x^{\alpha} \Rightarrow \underbrace{\ln y_i} = \ln\beta + \alpha \underbrace{\ln x_i}$: $\ln y$ is a linear function of $\ln x$

$A\vec{z} = \vec{b}$ $\quad \vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \ln\beta \\ \alpha \end{pmatrix}$ $\quad A = \begin{pmatrix} 1 & \ln x_1 \\ 1 & \ln x_2 \\ \vdots & \vdots \\ 1 & \ln x_m \end{pmatrix}$ $\quad \vec{b} = \begin{pmatrix} \ln y_1 \\ \ln y_2 \\ \vdots \\ \ln y_m \end{pmatrix}$

Solve: $\quad A^T A \vec{z} = A^T \vec{b}$ $\quad$ for $\vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \ln\beta \\ \alpha \end{pmatrix}$

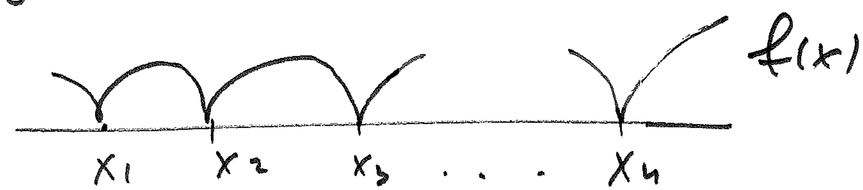
$$\Rightarrow \boxed{\beta = e^{z_1}, \quad \alpha = z_2}$$

Final Exam Review

- Exam is cumulative
- $\approx 2/3$ of exam problems will be on material since the midterm (starting from iterative methods)
- Notes: 1 sheet, double sided, letter size
- No calculators

HW #11
#1

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{j=1}^{n} c_j f(x_j) :$$ exact for polynomials of degree $\leq 2n-1$



$$f(x) = \prod_{i=1}^{n} (x-x_i)^2 :$$ polynom. of deg $2n$

Need to show that $\int_{-1}^{1} f(x)\,dx \neq \sum_{j=1}^{n} c_j f(x_j)$ in general.

x_j: roots of Legendre polynomia

$$f(x_j) = 0 \Rightarrow \sum_{j=1}^{n} c_j f(x_j) = 0$$

$$\int_{-1}^{1} \underbrace{f(x)}_{\geq 0}\,dx > 0 \Rightarrow \int_{-1}^{1} f\,dx > \sum_{j=1}^{n} c_j f(x_j)$$

- Review how to convert a higher order ODE into a system of 1st order ODEs.

HW #11 #5

$$y'' + \sin y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Let $u_1 = y$, $u_2 = y'$. Then $\qquad y' = f(y)$

$$\begin{cases} u_1' = u_2 \\ u_2' = -\sin(u_1) \end{cases} \qquad \begin{aligned} u_1(0) &= 1 = y(0) \\ u_2(0) &= 0 = y'(0) \end{aligned}$$

$$\Rightarrow \quad \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \vec{f}(\vec{u}) = \begin{pmatrix} u_2 \\ -\sin(u_1) \end{pmatrix} \quad \Rightarrow \quad \vec{u}' = \vec{f}(\vec{u})$$

- Review how to find upper bounds for integration quadratures, interpolation etc.
- Gram-Schmidt orthogonalization method
- Relation between local and global truncation errors.

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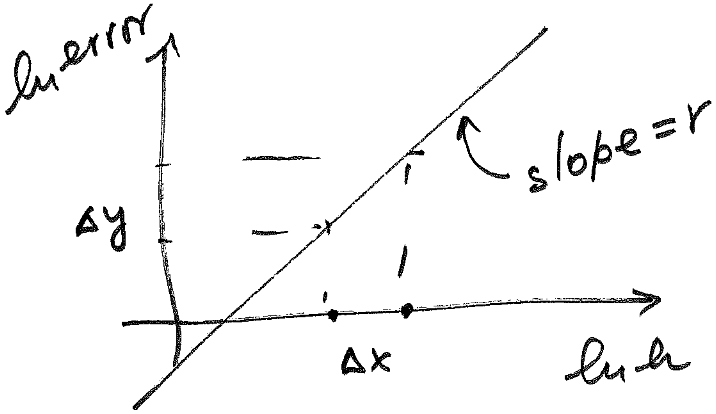
HW #10
#3

$\int_a^b \sin\sqrt{\pi x}\,dx$: analyze rates of convergence

error $= Ch^r$ or error $= Ce^{\alpha h}$

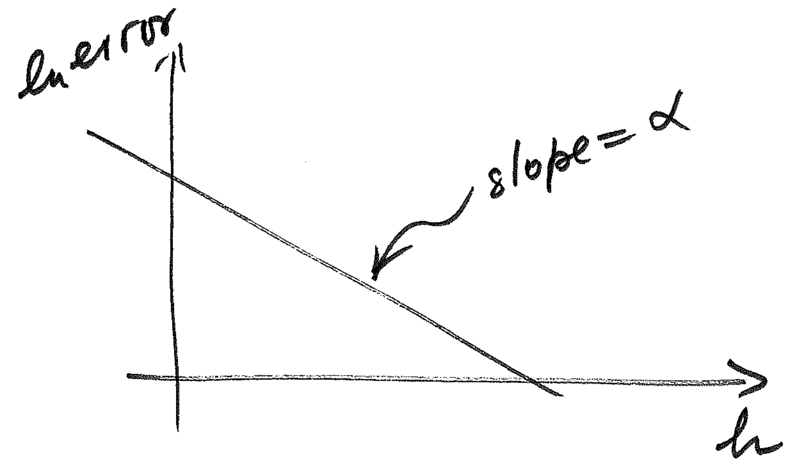
r: rate of convergence

$\underbrace{\ln \text{error}} = \ln C + r \underbrace{\ln h}$

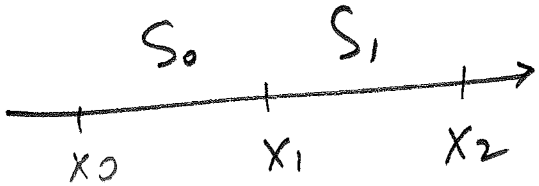


$\frac{\Delta y}{\Delta x} \approx r$

$\underbrace{\ln \text{error}} = \ln C + \alpha \underbrace{h}$



* How to construct cubic splines



find S_0, S_1

* Newton-Côtes and Gauss quadratures,
  Gauss-Legendre quadratures.
  Inner product, orthogonality.

* Polynomial interpolation, error bound
  Newton and Lagrange forms.

HW #8

#3

Show $\sum_{k=0}^{n} l_k(x) = 1$

$x_0 \quad x_1 \quad \ldots \quad x_n$

$n+1$ distinct points

$f(x) = p_n(x) + \text{error}$

$$\text{error} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\ldots(x-x_n)$$

$$p_n(x) = \sum_{k=0}^{n} f(x_k)\, l_k(x)$$

Let $f(x)=1$. $\Rightarrow$ $\underset{\substack{\| \\ 1}}{f(x)} = \overset{\substack{\sum_{k=0}^{n} \underbrace{f(x_k)}_{1} l_k(x) \\ \|}}{p_n(x)} + \text{error}$ $\Rightarrow$ $1 = \sum_{k=0}^{n} l_k(x) + \cancel{\text{error}}^{\,0}$

$f \equiv 1 \Rightarrow f^{(n+1)}(\xi) = 0 \Rightarrow \text{error} = 0$ $\Big\} \Rightarrow \sum_{k=0}^{n} l_k(x) = 1$

* Iterative methods: Jacobi, Gauss-Seidel, SOR
  convergence $\|B\|<1$ or $\rho(B)<1$
  A: positive definite, diagonally dominant, symmetric

* Nonlinear systems: Newton's method.

* Norms: vector and matrix norms, condition #

* Review lecture notes and HW problems.

* E'value / e'vector methods: power method, inverse iteration,
  Rayleigh quotient iteration

* Least squares problem