Lecture 4

Recall Taylor expansion (series) of $f(x)$ about $x=a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Taylor Thm:

$$f(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

where ξ is some value between a and x.

$$P_n(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n:$$ Taylor polynomial of degree n

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}:$$ remainder

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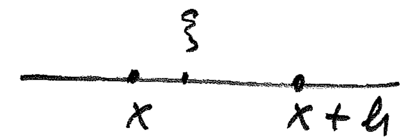
Ex $$f'(x) \sim \frac{f(x+h) - f(x)}{h} = D_+ f(x)$$



Error bound

$$f(x+h) = f(x) + f'(x)h + \frac{f''(\xi)}{2} h^2$$

$h = x - a$ where ξ is between x and $x+h$

$$\underbrace{\frac{f(x+h) - f(x)}{h}}_{\text{approximation}} = \underbrace{f'(x)}_{\text{exact}} + \frac{f''(\xi)}{2} \cdot h$$

$$\underbrace{\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right|}_{\text{error}} = \left| \frac{h}{2} f''(\xi) \right| \le \frac{h}{2} M$$

where $|f''(\xi)| \le M = \max |f''(\xi)|$

Def

Suppose $\lim_{h \to 0} F(h) = L$

If there exist constants p and C such that

$$|F(h) - L| \leq C \cdot h^p, \qquad C > 0$$

for sufficiently small h, we write this as

$$F(h) - L = O(h^p) \qquad \text{or} \qquad \frac{F(h) - L}{h^p} \sim \underset{C}{\text{const}}$$

The constant p is called <u>the order of accuracy</u>.
C: asymptotic constant. We also say that $F(h)$ converges to L with the <u>rate of convergence</u> $O(h^p)$.

Ex $\dfrac{f(x+h) - f(x)}{h} = f'(x) + O(h)$ w/ $p = 1$, $C = \frac{M}{2}$

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We wrote previously

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| = \left| \frac{h}{2} f''(\xi) \right| \leq \frac{M}{2} \cdot h$$

where $M = \max |f''(\xi)|$

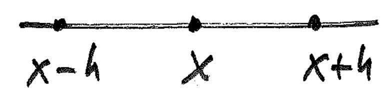
<u>Note</u> as $h \downarrow$, error also decreases.

as h decreases by a half, the error decreases approximately also by a half.

<u>Ex</u>

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h^2)$$

central difference approximation



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Proof

$$f(x+h) = f(x) + h\, f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(\xi_1)$$

ξ_1 is between x and $x+h$

$$f(x-h) = f(x) - h\, f'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f''(\xi_2)$$

ξ_2 is between x and $x-h$

Subtract.

$$f(x+h) - f(x-h) = 2h\, f'(x) + \frac{h^3}{6}\left(f'''(\xi_1) + f'''(\xi_2)\right)$$

$$\underbrace{\frac{f(x+h) - f(x-h)}{2h}}_{\text{approximation}} = \underbrace{f'(x)}_{\text{exact}} + \underbrace{\frac{h^2}{12}\left(f'''(\xi_1) + f'''(\xi_2)\right)}_{\text{error}}$$

error = exact - approximation

$$\left|\frac{f(x+h) - f(x-h)}{2h} - f'(x)\right| = \frac{h^2}{12}\left|f'''(\xi_1) + f'''(\xi_2)\right| \leq$$

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$$\le \frac{h^2}{12} \cdot 2\,|f'''(\xi)| = \frac{h^2}{6}\,|f'''(\xi)| \le \frac{h^2}{6} \cdot M$$

$$\therefore \left| \frac{f(x+h) - f(x-h)}{2h} - f'(x) \right| \le \frac{M}{6} h^2 \qquad \begin{array}{l} p=2 \\ C = \frac{M}{6} \end{array}$$

where $M = \max |f'''(\xi)|$

Note approximation $\dfrac{f(x+h)-f(x-h)}{2h}$ is 2^{nd} order accurate, whereas

$\dfrac{f(x+h)-f(x)}{h}$ is only 1^{st} order accurate.

Claim $\dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + O(h^2)$

Pf HW

Recall notation

$$D_+ f = \frac{f(x+h) - f(x)}{h}, \qquad D_- f = \frac{f(x) - f(x-h)}{h}$$

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = D_+ D_- f$$: can be shown

Back to rootfinding methods

<u>Note</u> Rootfinding methods are used to solve usually nonlinear equations.

<u>Thm</u> <u>Intermediate Value Thm</u>

Suppose $f(x)$ is continuous on $[a, b]$. Let K be any number between $f(a)$ and $f(b)$, ie.

$$f(a) < K < f(b) \quad \text{or} \quad f(b) < K < f(a)$$

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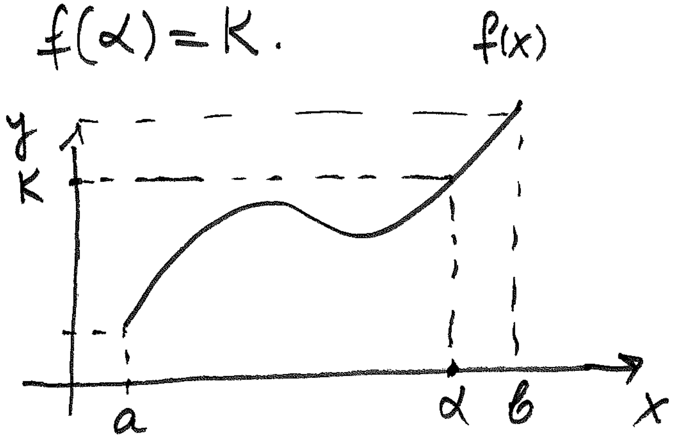
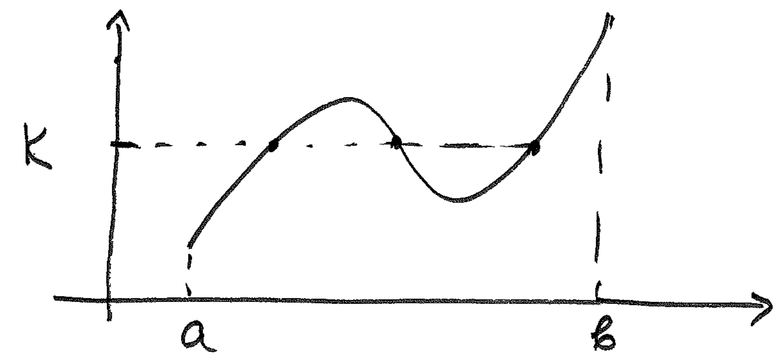
Then there exists a value $\alpha \in (a, b)$ such that $f(\alpha) = K$.



one value α



K is attained at three values of x

Bisection method uses this idea to find a root of $f(x) = 0 = K$

$$\left.\begin{array}{c} f(a) < 0 < f(b) \\ \text{or} \\ f(b) < 0 < f(a) \end{array}\right\} \Longleftrightarrow f(a) \cdot f(b) < 0$$

Then there exists $\alpha \in (a, b)$ such that $f(\alpha) = 0$.

Idea of bisection method:

check the sign of $f\left(\frac{a+b}{2}\right)$. Shrink the interval to subinterval that contains the root (values of f have opposite sign).

<u>Fixed-point iteration</u>

Suppose that $f(x)=0$ is equivalent to $x=g(x)$.

We say that α is a <u>fixed point</u> of g, $\alpha=g(\alpha) \Leftrightarrow$ α is a root of $f(x)$, i.e. $f(\alpha)=0$.

We define an iterative scheme

$$x_{n+1} = g(x_n) \quad \text{given } x_0$$

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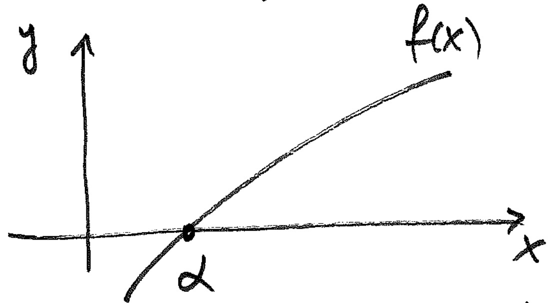
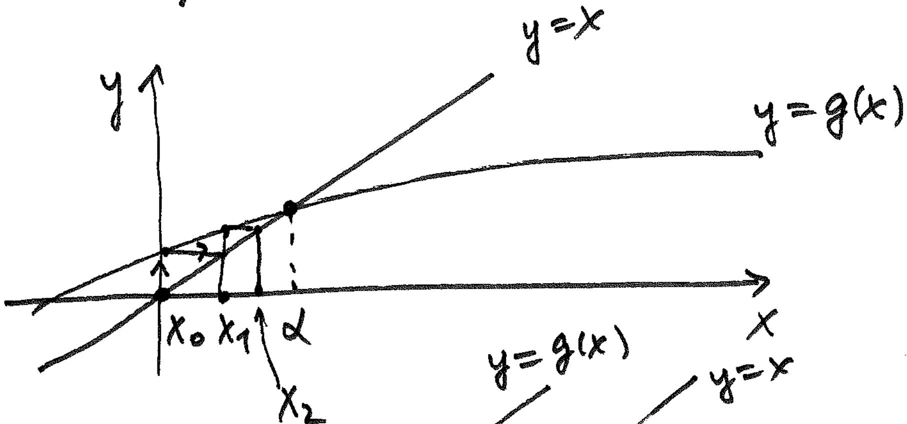
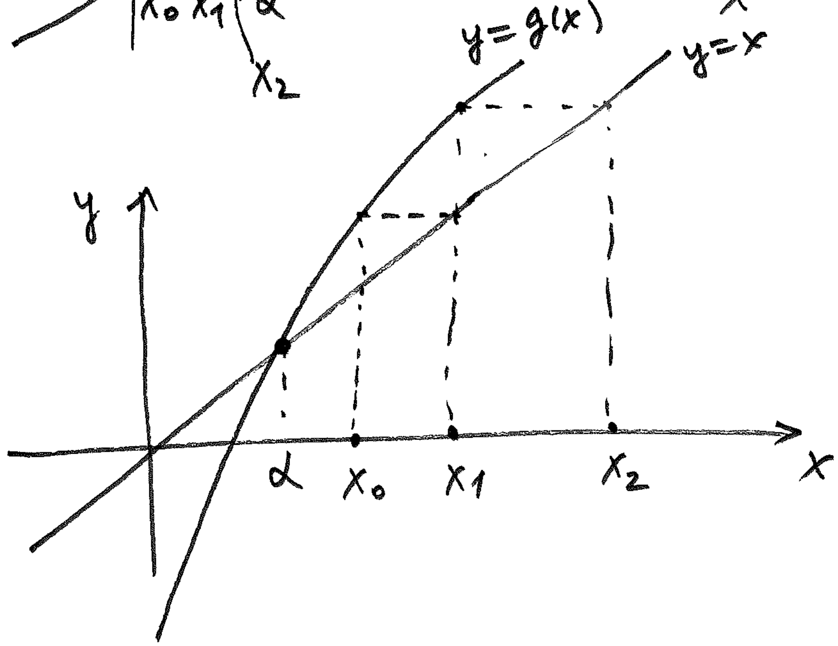


$f(x) = 0$



$x = g(x)$

converges



$x = g(x)$

diverges