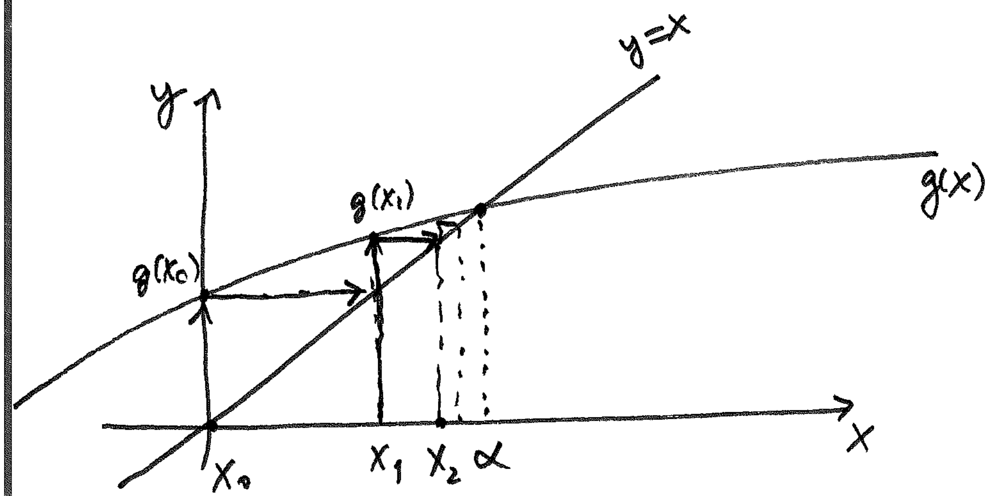
Lecture 5

University of Idaho

Fixed Point Iterations



$\boxed{x = g(x)}$

$x_{n+1} = g(x_n)$

$x_1 = g(x_0)$

$x_2 = g(x_1)$

sequence of $\{x_n\}$ converges to root α.

Ex $\quad f(x) = x^2 - 3 \qquad f(x) = 0 \Rightarrow x^2 - 3 = 0$

$\alpha = \sqrt{3} = \underline{\underline{1.73205}}$

$g_1(x) = x - \left(\frac{x^2-3}{2}\right) \qquad x = g_1(x)$

$-\frac{1}{2} \mid \quad x^2 - 3 = 0 \quad \Rightarrow \quad -\frac{1}{2}(x^2-3) = 0 \quad \mid +x$

$\underbrace{-\frac{1}{2}(x^2-3) + x}_{g_1(x)} = x$

n	x_n
0	1.5
1	1.875
2	1.617
3	1.810
4	1.672
5	1.774

converges

$$x^2 - 3 = 0 \Rightarrow x^2 = 3 \quad \Big| \ \frac{1}{x}$$

$$\Rightarrow x = \underbrace{\frac{3}{x}}_{g_2(x)}$$

$g_2(x) = \frac{3}{x}$

n	x_n
0	$1.5 = \frac{3}{2}$
1	2
2	1.5
3	2
4	1.5

diverges

Q What condition on $g(x)$ guarantees that $x_n \to \alpha$?

Thm (Existence and uniqueness of a fixed point)

A1. $g(x)$ maps $[a, b]$ into $[a, b]$, i.e. if $x \in [a, b]$
then $g(x) \in [a, b]$

A2. $g(x)$ is continuous on $[a, b]$

A3. $|g'(x)| \le k < 1$ for all $x \in [a, b]$

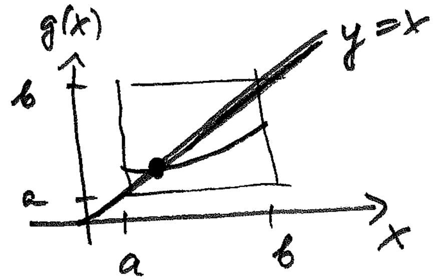
Note: A3 $\Rightarrow$ A2

1) A1 and A2 are satisfied $\Rightarrow$ $g(x)$ has a fixed point $\alpha \in [a, b]$

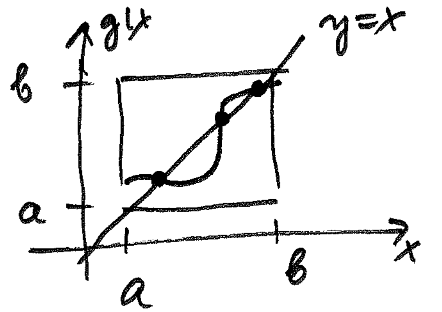
2) A1 and A3 are satisfied $\Rightarrow$ fixed point α is unique

University of Idaho

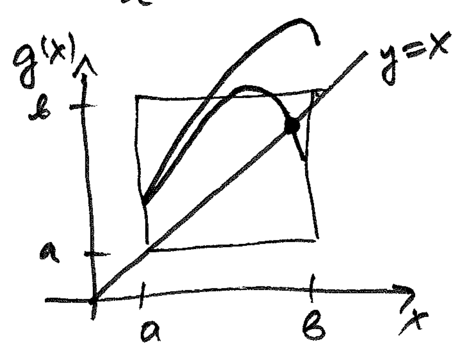
Note



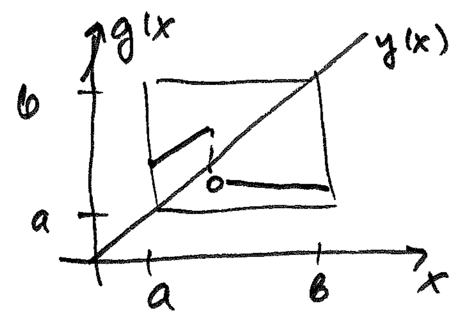
A_1, A_3 (and A_2) are satisfied
$\Rightarrow g(x)$ has a unique fixed pt



A_3 fails $\Rightarrow$ 3 fixed points
A_1 & A_2 hold



A_1 fails $\Rightarrow$ there may or may not be a fixed pt



A_2 fails, A_3 fails as well
$\Rightarrow$ there is no fixed pt
A_1 holds

University of Idaho

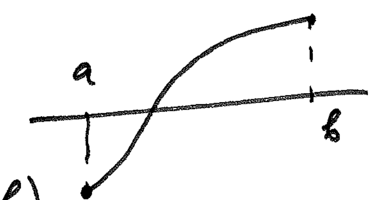
Proof of (1)

If $g(a)=a$ or $g(b)=b$, then we are done

Otherwise, we suppose that $a < g(x) < b$ for $\forall x \in [a,b]$ (all)

$a < g(a)$ and $g(b) < b$

Introduce $h(x) = x - g(x)$. $h(x)$ is continuous since $g(x)$ is continuous.



$h(a) = a - g(a) < 0$, $h(b) = b - g(b) > 0$

By Intermediate Value Thm, there is $\alpha \in (a,b)$ such that $h(\alpha)=0 \Rightarrow h(\alpha) = \alpha - g(\alpha) = 0$

$\Rightarrow \alpha = g(\alpha)$

$\therefore$ α is a fixed point of g in (a,b). ∎

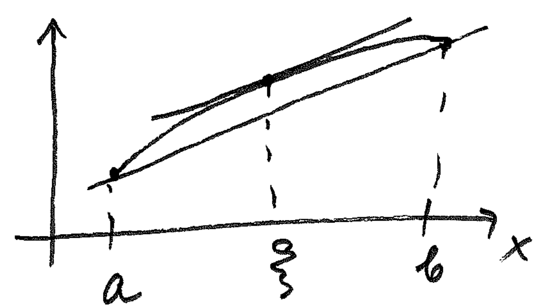
University of Idaho

Recall

Mean Value Thm

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a value $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b-a} \Rightarrow \boxed{f(b) - f(a) = f'(\xi)(b-a)}$$



tangent line of $f(x)$ at ξ has the same slope as secant line connecting $(a, f(a))$ and $(b, f(b))$

Proof of (2)

Suppose that α_1 and α_2 are two fixed points of $g(x)$ in $[a, b]$.

University of Idaho

Then

$$\underbrace{|\alpha_1 - \alpha_2|}_{\substack{g(\alpha_1) \quad g(\alpha_2)}} = |g(\alpha_1) - g(\alpha_2)| \overset{\text{Mean}}{\underset{\text{Value Thm}}{=}} |g'(\xi) \cdot (\alpha_1 - \alpha_2)| =$$

$$= |g'(\xi)| \cdot |\alpha_1 - \alpha_2| \leq K \cdot |\alpha_1 - \alpha_2|$$

$$\Rightarrow |\alpha_1 - \alpha_2| \leq K |\alpha_1 - \alpha_2|$$

$$\underbrace{(1-K)}_{\neq 0} |\alpha_1 - \alpha_2| \leq 0 \Rightarrow |\alpha_1 - \alpha_2| = 0 \Rightarrow \alpha_1 = \alpha_2 \ \lightning$$

$0 \leq K < 1 \Rightarrow 1 - K \neq 0$ $\qquad \therefore$ fixed pt is unique ∎

<u>Thm</u> (Convergence of fixed-point iterations)

A_1 and A_3 hold $\Rightarrow$ the sequence defined by $x_{n+1} = g(x_n)$ converges for any $x_0 \in [a, b]$.

Proof

$$|\alpha - x_{n+1}| = |g(\alpha) - g(x_n)| \overset{MVT}{=} |g'(\xi)| \cdot |\alpha - x_n| \leq K \cdot |\alpha - x_n|$$

$$\Rightarrow \underbrace{|\alpha - x_{n+1}|}_{\substack{\text{error at} \\ \text{iter. } n+1}} \leq K \cdot \underbrace{|\alpha - x_n|}_{\substack{\text{error at} \\ \text{previous iter. } n}}$$

$$\ldots \quad |\alpha - x_{n+1}| \leq K \cdot |\alpha - x_n| \leq K^2 |\alpha - x_{n-1}| \leq K^3 |\alpha - x_{n-2}| \leq \ldots$$

$$\ldots \leq K^{n+1} |\alpha - x_0|$$

as $n \to \infty$, $K^{n+1} \to 0$ since $K < 1$

$$\Rightarrow |\alpha - x_{n+1}| \leq \underset{\substack{\downarrow \\ 0}}{K^{n+1}} \underbrace{|\alpha - x_0|}_{\text{fixed}} \to 0 \quad \text{as } n \to \infty$$

$\Rightarrow x_{n+1} \to \alpha$ as $n \to \infty$ $\boxtimes$

<u>Def</u> of the order of convergence of a sequence

A sequence $\{x_n\}$ is said to converge to α with <u>order r</u> if there exists a constant C such that

$$|\alpha - x_{n+1}| \le C\,|\alpha - x_n|^r$$

r: order of convergence

<u>Note</u> This is equivalent to

$$|\alpha - x_n| \le C\,|\alpha - x_{n-1}|^r$$

or

$$\frac{|\alpha - x_n|}{|\alpha - x_{n-1}|^r} \le C$$