# Lecture 6



HW # 2

#1 $0.0012\,35 = 0.124 \times 10^{-2}$

$0.000005671 = 0.567 \times 10^{-5}$

$0.123 \times 10^{-2}$ $\qquad$ $0.123 \times 10^{-2} + 0.416 \times 10^{-2} = (0.123 + 0.416) \times 10^{-2}$

$0.416 \times 10^{-2}$ $\qquad$ $= 0.539 \times 10^{-2}$

$0.123 \times 10^{-2} + 0.459 \times 10^{-1} = 0.0123 \times 10^{-1} + 0.459 \times 10^{-1} = 0.471 \times 10^{-1}$

add mantissas, round to 3 digits

$$\begin{array}{r} 0.0123 \\ +\ 0.459 \\ \hline 0.4713 \end{array} \approx 0.471$$

#3 (a) composition of $D_+$ and $D_-$

(b) use Taylor Thm for $f(x+h)$, $f(x-h)$. Combine expansions and solve for $f''(x)$.

$$|\alpha - x_{n+1}| \le C\,|\alpha - x_n|^r$$

$r$: order of convergence

$C$: asymptotic constant

**Note**

1. $|\alpha - x_n| \le K\,|\alpha - x_{n-1}|$ : linear convergence
2. $K \sim |g'(\alpha)|$ : we want to choose the iteration function $g(x)$ in such a way that $|g'(\alpha)|$ is as small as possible

**Recall**

$f(x) = x^2 - 3$, $\quad \alpha = \sqrt{3}$

$g_1(x) = x - \dfrac{x^2-3}{2}$, $\quad g_1'(x) = 1 - \dfrac{2x}{2} = 1 - x$

$$|g_1'(\alpha)| = |g_1'(\sqrt{3})| = 1 - \sqrt{3} = 0.73 < 1$$

$g_2(x) = \dfrac{3}{x}$, $\quad g_2'(x) = -\dfrac{3}{x^2}$

$$|g_2'(\sqrt{3})| = \frac{3}{(\sqrt{3})^2} = 1$$

How do we determine the order of convergence numerically?

$\{x_n\} \to \alpha$ with order $r$ if

$$\underbrace{|\alpha - x_n|}_{\substack{\text{error at}\\ \text{iteration } n}} \leq C \, \underbrace{|\alpha - x_{n-1}|}_{\substack{\text{error at}\\ \text{iteration } n-1}}{}^{r}$$

Denote $E_n = |\alpha - x_n|$ : abs. error at iteration $n$.

$\Rightarrow \quad E_n \leq C\, E_{n-1}^{r} \quad$ or $\quad E_n \approx C\, E_{n-1}^{r} \quad \big| \ln$

$\ln(ab) = \ln a + \ln b$

$\ln a^n = n \ln a$

$\ln E_n \approx \ln(C\, E_{n-1}^{r})$

$\ln E_n \approx \ln C + r \ln E_{n-1}$

Assume that $C \approx 1 \Rightarrow \ln C \approx 0$

$$\Rightarrow \ln E_n \approx r \ln E_{n-1} \quad \Rightarrow \quad \boxed{r \approx \frac{\ln E_n}{\ln E_{n-1}}}$$

In practice, exact value $\alpha$ may not be known.
Can we still use this result to find order $r$ of convergence?

Yes, if we note that

$$\alpha - x_n \approx x_{n+1} - x_n \quad \Rightarrow \quad \boxed{\frac{\ln|x_{n+1}-x_n|}{\ln|x_n - x_{n-1}|} \approx r}$$

## <u>Newton's Method</u> (S 2.4)

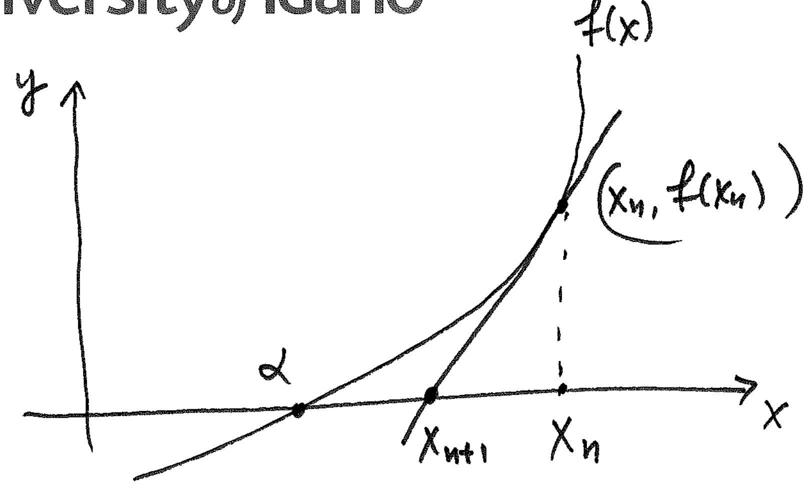
$f(x) = 0$

Idea: expand $f(x_{n+1})$ around the point $x = x_n$.

$$\cancelto{0}{f(\underbrace{x_{n+1}}_{x_n + (x_{n+1}-x_n)})} = f(x_n) + f'(x_n)(x_{n+1}-x_n) + \cancelto{0}{\cdots}$$

$\Rightarrow \quad 0 = f(x_n) + f'(x_n)(x_{n+1}-x_n)$. Then solve for $x_{n+1}$:

$$\Rightarrow \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{given } x_0}$$




Slope $f'(x_n) = \dfrac{f(x_n) - 0}{x_n - x_{n+1}}$ $\Rightarrow$ solve for $x_n$ as above

Ex $f(x) = x^2 - 3$

$$x_{n+1} = \underbrace{x_n - \frac{f(x_n)}{f'(x_n)}}_{g(x_n)} \qquad x_{n+1} = g(x_n)$$

$f' = 2x$

$$\Rightarrow \quad x_{n+1} = x_n - \frac{x_n^2 - 3}{2x_n}$$

$\Rightarrow$ iteration function $g(x) = x - \dfrac{f(x)}{f'(x)}$

| $n$ | $x_n$ | $f(x_n)$ | $\lvert\alpha - x_n\rvert$ |
|---|---|---|---|
| 0 | 1.5 | −0.75 | 0.232051 |
| 1 | 1.75 | 0.0625 | 0.017949 |
| 2 | 1.7321429 | 0.000319 | 0.000092 |
| 3 | 1.7320509 | 0.0000001 | 0.0000001 |

**Note**

1. $x_{n+1} = g(x_n)$ where $g(x) = x - \dfrac{f(x)}{f'(x)}$

If $f(\alpha) = 0$, $f'(\alpha) \neq 0$ ( $\alpha$ is a simple root of $f$ )

$$\Rightarrow \boxed{g'(\alpha) = 0, \quad g''(\alpha) \neq 0}$$

$$g(x) = x - \frac{f(x)}{f'(x)} \Rightarrow g'(x) = 1 - \frac{1}{[f'(x)]^2}\left[f'(x)\cdot f'(x) - f''(x)\cdot f(x)\right] =$$

$$= \frac{\cancel{(f'(x))^2} - \left(\cancel{(f'(x))^2} - f(x)\, f''(x)\right)}{(f'(x))^2} = \frac{f(x)\, f''(x)}{(f'(x))^2}$$

$$g'(\alpha) = \frac{\cancelto{0}{f(\alpha)}\, f''(\alpha)}{\underbrace{(f'(\alpha))^2}_{\neq 0}} = 0$$

One can show that $g''(\alpha) = \dfrac{f''(\alpha)}{f'(\alpha)} \neq 0$

2. It can be shown if $\alpha$ is a simple root of $f(x)$, then

$|\alpha - x_n| \leq C|\alpha - x_{n-1}|^2$ : <u>2nd order of convergence</u>

If $\alpha$ is a multiple root with multiplicity $m \geq 2$, then

$|\alpha - x_n| \leq C|\alpha - x_{n-1}|$ : <u>1st order of convergence</u>

3. Newton's method is more expensive than bisection, fixed-point (in general) since we have two function evaluations ($f(x_n)$, $f'(x_n)$) per iteration.

**Thm** Convergence of Newton's Method

Suppose function $f \in C^2[a,b]$ (i.e. $f$ has continuous second derivative) and assume that $f$ has a simple root $\alpha \in (a,b)$, i.e. $f(\alpha)=0$, $f'(\alpha) \neq 0$. Then Newton's Method converges to $\alpha$ if $x_0$ is chosen sufficiently close to $\alpha$.