Lecture 7

Proof

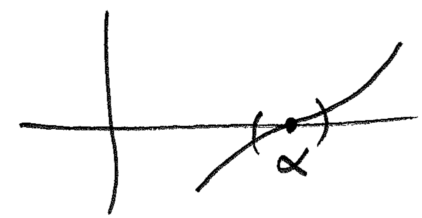
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow g(x) = x - \frac{f(x)}{f'(x)}$$

$f(\alpha)=0, \quad f'(\alpha)\neq 0$

We showed

$$g'(\alpha) = \frac{f(x)\, f''(x)}{(f'(x))^2}\Bigg|_{x=\alpha} = 0$$

Since $f(x)$, $f'(x)$, $f''(x)$ are continuous functions, $g'(x)$ is also continuous. Therefore, there exists a small interval containing $x=\alpha$ on which $|g'(x)| \le k < 1$, i.e. $g'(x)$ is close to 0, since $g'(x)\big|_{x=\alpha} = 0$.



By fixed point convergence Thm, sequence $\{x_n\}$ of iterations obtained using Newton's method with $g(x) = x - \frac{f(x)}{f'(x)}$ will converge to α if x_0 is chosen sufficiently close to α, i.e. x_0 has to inside that small interval. ∎

Thm Order of convergence of Newton's Method

Suppose that $f \in C^2[a,b]$, $f(\alpha)$, $f'(\alpha) \neq 0$, x_0 is sufficiently close to α. Then Newton's method converges quadratically, i.e.

$$|\alpha - x_{n+1}| \leq c\,|\alpha - x_n|^2$$

Proof

Expand $f(x)$ in the neighborhood of $x = x_n$.

$$0 = f(\alpha) = f(x_n) + f'(x_n)\cdot(\alpha - x_n) + \frac{f''(\xi)}{2!}(\alpha - x_n)^2 \quad \Big| \frac{1}{f'(x_n)}$$

$$\alpha = x_n + (\alpha - x_n)$$

ξ is between α and x_n

$$0 = \frac{f(x_n)}{f'(x_n)} + (\alpha - x_n) + \frac{f''(\xi)}{2f'(x_n)}(\alpha - x_n)^2$$

Newton's method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$\Rightarrow \frac{f(x_n)}{f'(x_n)} = x_n - x_{n+1}$$

$$0 = (\cancel{x_n} - x_{n+1}) + (\alpha - \cancel{x_n}) + \frac{f''(\xi)}{2f'(x_n)} (\alpha - x_n)^2$$

$$0 = (\alpha - x_{n+1}) + \frac{f''(\xi)}{2f'(x_n)} (\alpha - x_n)^2$$

$$\therefore \quad \alpha - x_{n+1} = - \frac{f''(\xi)}{2f'(x_n)} (\alpha - x_n)^2$$

$$\boxed{|\alpha - x_{n+1}| \le C \cdot |\alpha - x_n|^2,} \quad \text{where} \quad C = \max_{a \le x \le b} \frac{|f''(\xi)|}{2|f'(x)|}$$

$$\text{or} \quad C = \frac{\max |f''(\xi)|}{2 \cdot \min |f'(x)|}$$

Note

We showed that

$$\underset{\sim\sim}{\alpha - x_{n+1}} = -\frac{1}{2} \frac{f''(\xi)}{f'(x_n)} (\alpha - x_n)^2 \approx -\frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)} (\alpha - x_n)^2 = \underset{\sim\sim}{M(\alpha - x_n)^2}$$

provided $f'(\alpha) \neq 0$

where $\boxed{M = -\frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)}}$

$$\Rightarrow \quad \alpha - x_{n+1} \approx M(\alpha - x_n)^2 \quad \Big| \cdot M$$

$$M(\alpha - x_{n+1}) \approx \left[M(\alpha - x_n)\right]^2 = \left[M(\alpha - x_{n-1})\right]^{2^2} = \dots = \left[M(\alpha - x_0)\right]^{2^{n+1}}$$

α: root, $\quad x_0$: initial guess, $\quad M$: const

For convergence we need $|M(\alpha - x_0)| < 1$.

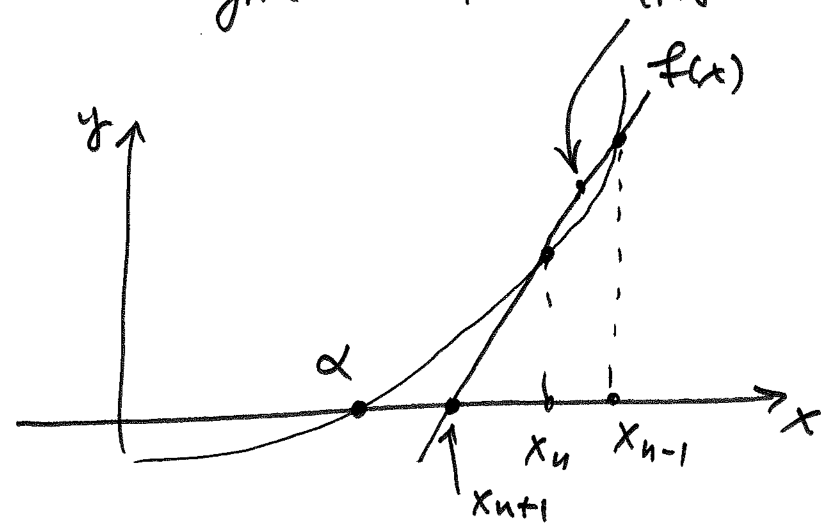
Note if $|M|$ is large, then x_0 may need to be very close to α for convergence.

Secant Method

$$x_{n+1} = x_n - \frac{f(x_n)}{\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}$$

given x_0, x_1 $\quad$ (x,y): arbitrary pt on secant line

Geometrically:



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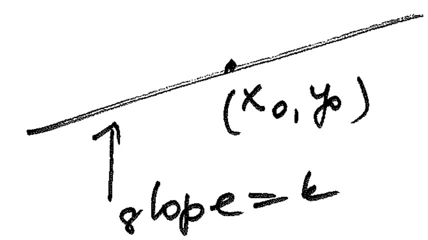
Slope of secant line : $\dfrac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}$

Equation of secant line

$$\frac{y-f(x_n)}{x-x_n} = \frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}$$

$y - y_0 = k(x - x_0)$



Set $y=0$ and $x = x_{n+1}$ and solve for x_{n+1} : as above

Note

1. It can be shown that $|\alpha - x_n| \le C \cdot |\alpha - x_{n-1}|^p$: superlinear convergence

where $p = \dfrac{1+\sqrt{5}}{2} \sim 1.6$: golden ratio

2. Secant method converges more slowly than Newton's method but faster than fixed-point iteration (in general).

3. Secant method has 1 function evaluation per iteration.

Summary

1. Bisection
   - linear convergence
   - guaranteed to converge
   - do not need $f'(x)$
2. Secant method
   - superlinear convergence
   - convergence depends on initial guesses x_0 and x_1
   - do not need $f'(x)$
3. Newton's method
   - converges quadratically (if root α is simple)
   - convergence depends on x_0
   - need to know $f'(x)$

Linear Systems of Equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\text{— — — — —}$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

n equations, n unknowns $x_1, x_2, .., x_n$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1, .., n$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{b}$$

a_{ij} : i^{th} row, j^{th} column

$\Rightarrow$ $Ax = b$, $\quad A = (a_{ij})$

<u>Thm</u>

Let A be an $n \times n$ matrix. The following statements are equivalent.

1. The equation $Ax = b$ has a unique solution for any b.
2. $\det A \neq 0$
3. Matrix A is invertible