Math 432: midterm review

10/10/2013

Conditioning of roots of a polynomial

$p(x)$: polynomial

Ex Wilkinson polynomial is

$$p(x) = (x-1)(x-2) \dots (x-20) =$$

$$= x^{20} - 210x^{19} + \dots$$

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

nth degree polynomial

$a_0, a_1, \dots, a_n$: coefficients, $a_n \neq 0$

x_j, $j = 1, \dots, n$: roots of $p(x)$

Here $x_j = j$: all distinct in this case

$$\text{cond}_j = \frac{|a_i x_j^{i-1}|}{|p'(x_j)|}$$

cond # of root x_j wrt to perturbation of coefficient a_i

Ex: perturb coefficient $a_{19} = -210 + \varepsilon$

we want to compute cond_{16} (cond # for root $x_{16} = 16$)

$$\text{cond}_{16} = \frac{|-210 \cdot 16^{19-1}|}{|p'(16)|}$$

$$i = 19, \; j = 16$$

HW # 4, problem # 1

$$p(x) = (x-1)(x-0.99)(x-2)$$

$$x_1 = 1, \quad x_2 = 0.99, \quad x_3 = 2$$

since roots x_1, x_2 are close, they may be ill-conditioned.

$$p'(x) = (x-0.99)(x-2) + (x-1)\left(2x - 2 - 0.99\right)$$

$$p(x) = (x-1)(x-2)(x-0.99) = (x^2-3x+2)(x-0.99) =$$

$$= x^3 - 0.99\,x^2 - 3x^2 + 3(0.99)x + 2x - 2(0.99)$$

$$= x^3 - 3.99\,x^2 + [3(0.99)+2]x - 2(0.99)$$

$$a_3 = 1 \quad a_2 = -3.99 \quad a_1 = 3(0.99+2), \; a_0 = -2(0.99)$$

We need to verify if cond_1 and/or cond_2 are large.

$$\text{cond}_1 = \frac{|a_i x_1^{i-1}|}{|p'(x_1)|} \qquad i = 0, .., 3$$

and

$$\text{cond}_2 = \frac{|a_i x_2^{i-1}|}{|p'(x_2)|} \qquad i = 0, .., 3$$

until you get large cond #

For example, $i = 2 \Rightarrow a_2 = -3.99$

$$\text{cond}_1 = \frac{|a_2 x_1^{2-1}|}{|p'(x_1)|} = \frac{3.99 \cdot 1}{10^{-2}} \approx 400$$

$$p'(x_1) = (x_1 - 0.99)(x_1 - 2) + (x_1 - 1)(2x_1 - 2.99)$$

$$= (1-0.99)(1-2) + \cancelto{0}{(1-1)\qquad} =$$

$$= -0.01$$

HW #6 problem # 4

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

We use partial pivoting.
Interchange rows 1 and 3 $\Rightarrow$

$$P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{pmatrix} \qquad m_{21} = -\frac{1}{2}, \quad m_{31} = 0$$

$$\sim \begin{pmatrix} 2 & 0 & 1 \\ 0 & \boxed{1} & \frac{1}{2} \\ 0 & \boxed{2} & -1 \end{pmatrix}$$

$$1 \leftarrow 1 + \left(-\frac{1}{2}\right) \cdot 1 = \frac{1}{2}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ +\frac{1}{2} \updownarrow & 1 & 0 \\ 0 & * & 1 \end{pmatrix}$$

Pivoting: interchange rows 2 and 3

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \qquad m_{32} = -\frac{1}{2}$$

Gaussian elimination:

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \equiv U$$

$$\frac{1}{2} + \frac{1}{2} = 1$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Check: $PA = LU$

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \to \overset{U \swarrow}{\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}} \qquad p = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$L \rightarrow$

permutation vector

To solve $Ax = b$, we multiply both sides by P:

$$\underbrace{PA}_{LU}x = Pb \quad \Rightarrow \quad L\underbrace{Ux}_{y} = Pb$$

$$Pb = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

<u>Step 2</u>: Solve $Ly = Pb$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

$y_1 = 0 \qquad y_2 = -2$

$$\frac{1}{2} \cdot \cancelto{0}{y_1} + \frac{1}{2} \cdot \underbrace{y_2}_{-2} + y_3 = 1 \quad \Rightarrow \quad -1 + y_3 = 1$$

$$\Rightarrow y_3 = 2$$

$$y = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} \quad \checkmark$$

<u>Step 3</u>: Solve $Ux = y$

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$$

$x_3 = 2$

$x_2 = (-2 + 1\cdot \underset{\overset{\shortparallel}{2}}{x_3})/2 = 0 \qquad \Rightarrow x = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

$2\cdot x_1 + 1\cdot \underset{\overset{\shortparallel}{2}}{x_3} = 0 \Rightarrow x_1 = -1$

<u>Check:</u> $Ax = b$ ✓

$\boxed{\#2c}$ $A = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix}$

Compute $A^{-1} = \dots$

Compute $\|A\|$, $\|A^{-1}\|$, $\mathrm{Cond}(A) = \|A\|\cdot\|A^{-1}\|$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\boxed{\text{HW \#2, problem \#5}}$

$\rho(A) < 1 \Rightarrow I - A$ is nonsingular

↗ Assume that $I - A$ is singular.

A has e'value λ $\Rightarrow$ $A - \lambda I$ is singular

$I - A$ is singular $\Rightarrow$ $\lambda = 1$ is e'value of A

$$\rho(A) = \max_{\lambda \text{ is e'value of } A} |\lambda| \quad \Rightarrow \quad \rho(A) \geq 1 \quad ↯$$