Position Systems

$$x = \pm(a_n a_{n-1} \dots a_1 a_0 . a_{-1} a_{-2} \dots)_B =$$

$$= \pm\left(a_n \cdot B^n + a_{n-1} \cdot B^{n-1} + \dots + a_1 \cdot B + a_0 + a_{-1} \cdot B^{-1} + a_{-2} \cdot B^{-2} + \dots\right)$$

B: base; a_i: digits, $0 \le a_i \le B-1$, $a_n \neq 0$

Ex $(\underset{a_4}{1}\underset{a_3}{0}\underset{a_2}{1}\underset{a_1}{0}\underset{a_0}{1}.\underset{a_{-1}}{0}\underset{a_{-2}}{1})_2 = 1 \cdot 2^4 + 1 \cdot 2^2 + 1 \cdot 2^0 + 1 \cdot 2^{-2}$

$= 16 + 4 + 1 + \frac{1}{4} = (21.25)_{10}$

$B = 2$: binary system

$B = 10$: decimal system

Floating point numbers

$$x = \pm(0.a_{-1} a_{-2} \dots a_{-n}) \cdot B^e$$

$0.a_{-1} a_{-2} \dots a_{-n}$: mantissa $\quad a_{-1} \neq 0$

e: exponent $\quad -M \le e \le M$

Note $(10101.01)_2 = \underbrace{0.1010101}_{\text{mantissa}} \cdot 2^5 \quad e = 5$

Ex $n=4$, $B=2$, $M=3$

$$x_{min} = (0.1000)_2 \cdot 2^{-3} = \frac{1}{2} \cdot 2^{-3} = 2^{-4} = (0.0625)_{10}$$

$$x_{max} = (0.1111)_2 \cdot 2^3 = (111.1)_2 = 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 +$$

$$+ 1 \cdot \frac{1}{2} = 4 + 2 + 1 + \frac{1}{2} = (7.5)_{10}$$

Note Most computers have two values of n, for single and double precision arithmetic.

format long

x: real number
$fl(x)$: floating point representation of x
$x - fl(x)$: roundoff error

Ex $x = 3.1428$ Due to roundoff error, we can
$y = 3.1417$ assume that these numbers have only 4 significant digits

Consider $x - y = 0.0011 = 0.1100 \cdot 10^{-2}$: only one significant digit

significant

This is called loss of significance due to cancellation of digits.

Ex $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For $h > 0$ we can approximate

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} = D_+ f$$

Recall Taylor series expansions of $f(x)$ around $x=a$

$$f(x) = f(a) + f'(a) \cdot (x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

$$a + \underbrace{(x-a)}_{\Delta x}$$

Taylor Thm:

$$f(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n +$$

$$+ \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

where ξ is some value between a and x

$$P_n(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n :$$

Taylor polynomial of order n

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} : \text{remainder}$$