

1/13/2010

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Position Systems

$$x = \pm (a_n a_{n-1} \dots a_1 a_0 . a_{-1} a_{-2} \dots)_B = \\ = \pm (a_n \cdot B^n + a_{n-1} \cdot B^{n-1} + \dots + a_1 \cdot B + a_0 \\ + a_{-1} \cdot B^{-1} + a_{-2} \cdot B^{-2} + \dots)$$

B : base; a_i : digits, $0 \leq a_i \leq B-1$, $a_n \neq 0$

Ex

$$\begin{aligned} (10101.01)_2 &= 1 \cdot 2^4 + 1 \cdot 2^2 + 1 \cdot 2^0 + 1 \cdot 2^{-2} \\ &= 16 + 4 + 1 + \frac{1}{4} = (21.25)_{10} \end{aligned}$$

$B=2$: binary system

$B=10$: decimal system

Floating point numbers

$$x = \pm (0.a_1 a_2 \dots a_{-n}) \cdot B^e$$

$0.a_1 a_2 \dots a_{-n}$: mantissa $a_1 \neq 0$

e : exponent $-M \leq e \leq M$

Note

$$(10101.01)_2 = \underbrace{0.1010101}_{\text{mantissa}} \cdot 2^5 \quad e=5$$

Ex $n=4, B=2, M=3$

$$x_{\min} = (0.1000)_2 \cdot 2^{-3} = \frac{1}{2} \cdot 2^{-3} = 2^{-4} = (0.0625)_{10}$$

$$x_{\max} = (0.1111)_2 \cdot 2^3 = (111.1)_2 = 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 + 1 \cdot \frac{1}{2} = 4 + 2 + 1 + \frac{1}{2} = (7.5)_{10}$$

Note Most computers have two values of n , for single and double precision arithmetic.

format long

x : real number

$fl(x)$: floating point representation of x

$x - fl(x)$: roundoff error

Ex $x = \boxed{3.1428}$
 $y = \boxed{3.1417}$

Due to roundoff error, we can assume that these numbers have only 4 significant digits

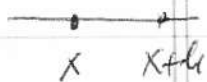
Consider $x - y = 0.0011 = 0.\overset{\text{significant}}{\overbrace{1100}} \cdot 10^{-2}$; only one significant digit

This is called loss of significance due to cancellation of digits.

$$\underline{\text{Ex}} \quad f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

For $h > 0$ we can approximate

$$f'(x) \sim \frac{f(x+h) - f(x)}{h} = D_+ f$$



Recall Taylor series expansion of $f(x)$ around $x=a$

$$f(x) = f(a) + f'(a) \cdot (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots$$

$$a + \underbrace{(x-a)}_{\Delta x}$$

Taylor Thm:

$$f(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}$$

where ξ is some value between a and x

$$P_n(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Taylor polynomial of order n

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1} : \text{remainder}$$