1/15/2010

Error Bound

$$f(x+h) = f(x) + h \cdot f'(x) + \frac{h^2}{2} f''(\xi)$$ where ξ is between x and $x+h$

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| = \left| \frac{h}{2} f''(\xi) \right| \le \frac{h}{2} M$$

where $M = \max |f''(\xi)|$

Ex $f(x) = e^x$, $f'(x) = e^x$

$$f' \sim \frac{f(x+h) - f(x)}{h} = D_+ f \qquad x = 1$$

$$f' \sim \frac{e^{x+h} - e^x}{h}$$

What happens with error between f' and its approximation as $h \searrow$

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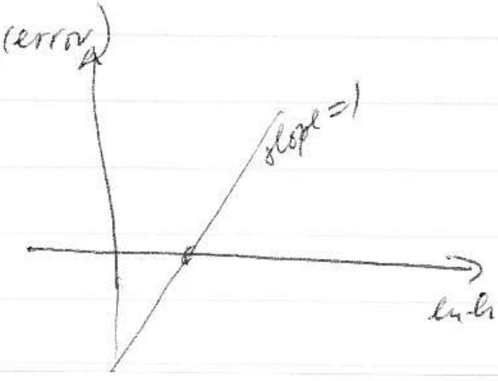
error $\sim h \cdot C$ where $C = \frac{M}{2}$

$$\underbrace{\ln(\text{error})}_{y} \cong \underbrace{\ln h}_{x} + \ln C$$



Def

Suppose $\lim_{h\to 0} F(h) = L$

If there exist constants p and C such that

$$\left| F(h) - L \right| \leq C \cdot h^p \qquad C>0$$

for sufficiently small h, we write this as

$$F(h) - L = O(h^p) \quad \text{or} \quad \frac{F(h)-L}{h^p} \sim \text{const}$$

The constant p is called order of accuracy.
We also can say that $F(h)$ converges to L
with rate of convergence $O(h^p)$

(1) Ex $\quad \dfrac{f(x+h) - f(x)}{h} = f'(x) + O(h) \quad$ with $p=1$, $C = \frac{M}{2}$

since
error $= \frac{h}{2} f''(\xi)$ $\leq \frac{h}{2} M$ We wrote previously

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| = \left| \frac{h}{2} f''(\xi) \right| \leq \underbrace{\frac{h}{2} M}_{= C \cdot h^p}$$

where $M = \max |f''|$

Note As $h\downarrow$, error also decreases.
As h decreases by a half, the error decrease approximately also by a half.

(2) Ex $\dfrac{f(x+h)-f(x-h)}{2h} = D_0 f = f'(x) + O(h^2)$, $p=2$

$C = \dfrac{\max|f'''|}{6}$

$x-h \quad x \quad x+h$

central approximation

proof

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(\xi_1)$$

$$-\quad f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(\xi_2)$$

subtract

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{6}\left(f'''(\xi_1) + f'''(\xi_2)\right)$$

$$\underbrace{\frac{f(x+h) - f(x-h)}{2h}}_{\text{approx.}} = \underbrace{f'(x)}_{\text{exact}} + \underbrace{\frac{h^2}{12}\left(f'''(\xi_1) + f'''(\xi_2)\right)}_{\text{error}}$$

$$\left|\frac{f(x+h) - f(x-h)}{2h} - f'(x)\right| \le \underbrace{\frac{h^2}{12}\left|f'''(\xi_1) + f'''(\xi_2)\right|}$$

$$\approx \frac{h^2}{12}\cdot 2\left|f'''(\xi)\right|$$

$$= \frac{h^2}{6}\left|f'''(\xi)\right|$$

Note Approximation $\dfrac{f(x+h)-f(x)}{h}$ is 1st order accurate, whereas

$\dfrac{f(x+h)-f(x-h)}{2h}$ is 2nd order accurate.

Taylor series:

$$f(x+h) = f(x) + h \cdot f'(x) + \frac{h^2}{2!} f''(x) + \dots$$

$$\Rightarrow \underbrace{f'(x)}_{\text{exact}} = \underbrace{\frac{f(x+h) - f(x)}{h}}_{\text{approximation}} \underbrace{- \frac{h}{2} f''(x) + \dots}_{\text{error (truncation error)}}$$

Taylor Thm

$$f(x+h) = \underbrace{f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \dots + \frac{h^n}{n!} f^{(n)}(x)}_{P_n(x)} + \underbrace{\frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi)}_{R_n(x)}$$

$$\underbrace{f(x+h)}_{\text{exact}} = \underbrace{P_n(x)}_{\text{approximation}} + \underbrace{R_n(x)}_{\text{error}}$$

$P_n(x)$: n^{th} order Taylor polynomial

$R_n(x)$: remainder

error = exact − approx.

X: exact

$\tilde{X}$: approximation

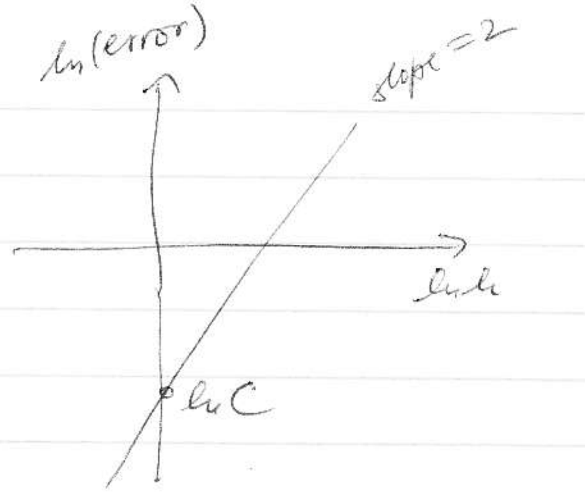
$X - \tilde{X}$: error (absolute error)

$\frac{X - \tilde{X}}{X}$: relative error

$\frac{X - \tilde{X}}{X} \cdot 100\%$: error in percentage

$$\text{error} = C \cdot h^2$$

$$\ln(\text{error}) = \ln C + \underset{\text{slope}}{2} \ln h$$



<u>Claim</u> $$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + O(h^2)$$

<u>Pf</u> HW

<u>Notation</u>

$$\frac{f(x+h) - f(x)}{h} = D_+ f$$

$$\frac{f(x) - f(x-h)}{h} = D_- f$$

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = D_+ D_- f$$

<u>Check</u> $$D_+ D_- f = D_+ (D_- f) = D_+ \left(\frac{f(x) - f(x-h)}{h} \right) =$$

$$= \frac{1}{h} D_+ \left(f(x) - f(x-h) \right) = \frac{1}{h} \left[\frac{f(x+h) - f(x)}{h} - \right.$$

$x \to x-h$ $$\left. - \frac{f(x-h+h) - f(x-h)}{h} \right] = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$