1/22/2010

HW #1: please also read Sections 1.2 – 1.4

Ex about truncation error

How many terms/iterations are needed to guarantee that the error of the approximation does not exceed a given tolerance?

We will consider this using an example of Taylor expansion.

Let $f(x) = e^x$. Expand it about $x=0$.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots +$$

$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

where ξ is a pt between x and a.

$a=0$

$$f(x) = f(0) + f'(0)x + \dots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$$

We would like to use Taylor polynomial

$$P_n(x) = f(0) + f'(0)x + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

to approximate $f(x)$.

Q How many terms one should keep to guarantee that the (absolute) error is < 0.01 on $[-1, 2]$?

$$f(x) = e^x, \quad f'(x) = \dots = f^{(n)}(x) = e^x$$

$$e^x = e^0 + e^0 \cdot x + \frac{e^0}{2!} x^2 + \dots + \frac{e^0}{n!} x^n + \frac{e^\xi}{(n+1)!} x^{n+1}$$

$$\underset{\text{exact}}{e^x} = \underbrace{1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}}_{\substack{P_n(x) \\ \text{approximation}}} + \underbrace{\frac{e^\xi}{(n+1)!} x^{n+1}}_{\text{error}}$$

$$\text{error} = |e^x - P_n(x)| = \left| \frac{e^\xi}{(n+1)!} x^{n+1} \right|$$

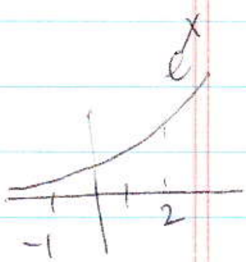
where ξ is between 0 and x.

We want:

$$\left| \frac{e^\xi}{(n+1)!} x^{n+1} \right| < 0.01 \quad \text{for all } x \in [-1, 2]$$

Note that

$$\left| \frac{e^\xi}{(n+1)!} x^{n+1} \right| \le \max_{\xi, x} \left| \frac{e^\xi}{(n+1)!} x^{n+1} \right| \le$$



$$\le \max_{\xi \in [-1,2]} \left| \frac{e^\xi}{(n+1)!} \right| \cdot \max_{x \in [-1,2]} \left| x^{n+1} \right| = \underbrace{\frac{e^2}{(n+1)!} \cdot 2^{n+1}}_{\text{max possible error}}$$

If we find n such that max error is less than 0.01 then this will guarantee that the actual error will be < 0.01 for all x

$$\frac{e^2}{(n+1)!} 2^{n+1} < 0.01 \quad \Rightarrow \quad n \geq ?$$

$n=5$: $\frac{e^2}{6!}2^6 = 0.6568 \nless 0.01$

$n=7$: $\frac{e^2}{7!}2^8 = 0.0469 \nless 0.01$

$n=8$: $\frac{e^2}{9!}2^9 = 0.0104 \nless 0.01$

$n=9$: $\frac{e^2}{10!}2^{10} = 0.002085 < 0.01$

$\therefore$ $\boxed{n \geq 9}$

Rootfinding Methods (Chapter 2)

Rootfinding methods are used to solve usually nonlinear equations.

Given a function $f(x)$, α is a root of $f(x)$ if $f(\alpha)=0$

Ex $f(x) = x^2 - 3x + 2$. Roots are $\alpha = 1, 2$

Thm Intermediate Value Theorem

Suppose $f(x)$ is continuous on $[a,b]$. Let K be any number between $f(a)$ and $f(b)$, i.e.

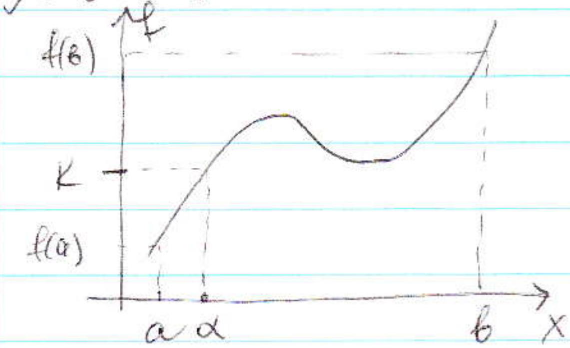
$$f(a) < K < f(b)$$

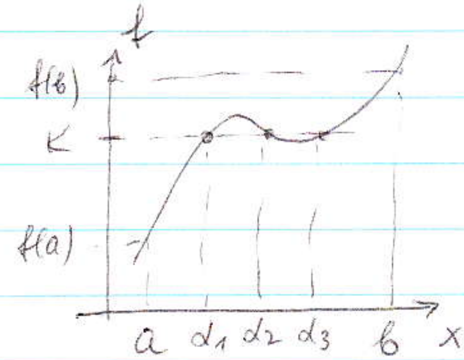
or

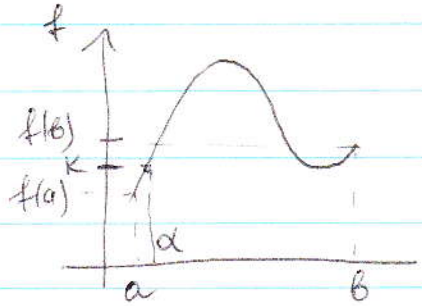
$$f(b) < K < f(a)$$

Then there exists a value $\alpha \in (a,b)$ such that

$f(\alpha) = K$







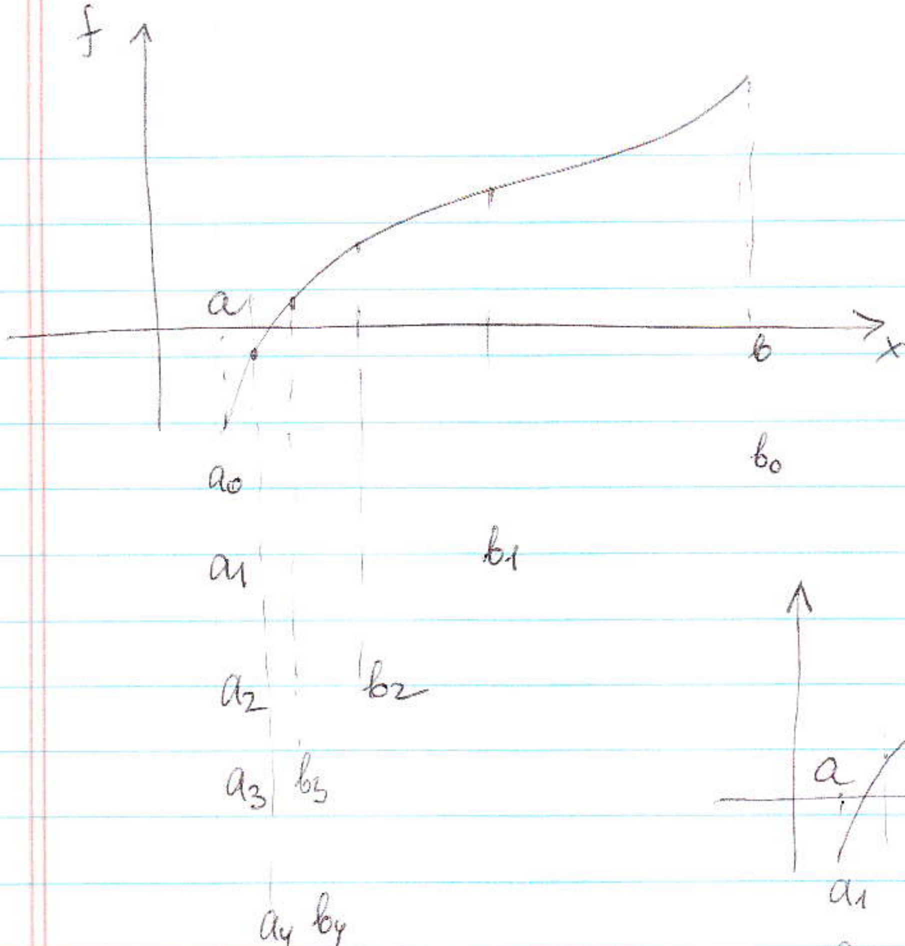
Application: $f(x) = 0$

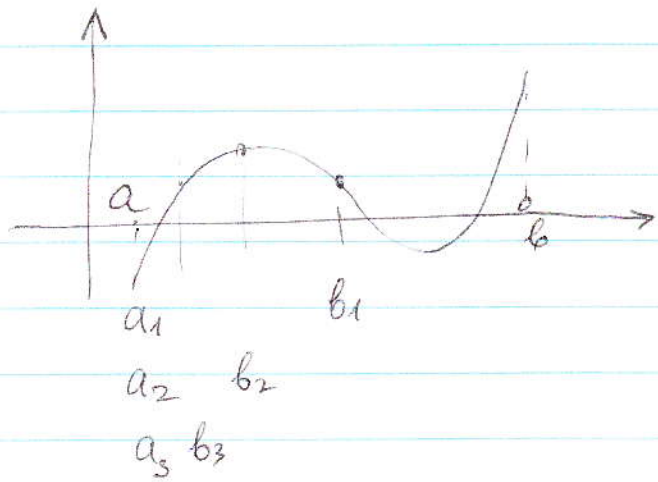
$$\left.\begin{array}{l} f(a) < 0 < f(b) \\ \text{or } f(b) < 0 < f(a) \end{array}\right\} \Leftrightarrow f(a)\cdot f(b) < 0$$

Then there exists $\alpha \in (a,b)$ such that $f(\alpha) = 0$.

Idea

Check the sign of $f\left(\frac{a+b}{2}\right)$. Shrink the interval that contains the root.





Bisection method

(assume $f(a) \cdot f(b) < 0$)

$a_0 = a, \quad b_0 = b$

$n = 0$

$x_n = (a_n + b_n)/2$

if $f(a_n) \cdot f(x_n) < 0$ then

$a_{n+1} = a_n$

$b_{n+1} = x_n$

else

$a_{n+1} = x_n, \quad b_{n+1} = b_n$

endif

$n = n+1$

Ex $f(x) = x^2 - 3, \quad x \in [1,2]$

$f(1) = -2 < 0, \quad f(2) = 1 > 0 \Rightarrow$ there exists a root $\alpha \in (1,2)$

$$\alpha = \sqrt{3} = 1.73205$$

n	a_n	b_n	x_n	$f(x_n)$	$\lvert\alpha - x_n\rvert$
0	1	2	1.5	−0.75	0.2321
1	1.5	2	1.75	0.0625 (!)	0.0179
2	1.5	1.75	1.625	−0.3594	0.1071
3	1.625	1.75	1.6875	−0.1523	0.0446
4	1.6875	1.75	1.71875	−0.0459	0.0133