1/25/2010

(#3 f) $f(x) = \ln(x + \sqrt{x^2+1})$

If x is large and negative
$\downarrow$
$$\sqrt{x^2+1} \approx \sqrt{x^2} = x$$

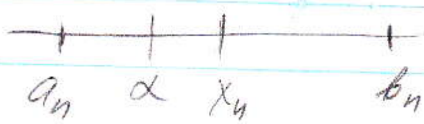
In class example: $\sqrt{x+1} - \sqrt{x} \longrightarrow x + \sqrt{x^2+1} = \dfrac{(x+\sqrt{x^2+1})(x-\sqrt{x^2+1})}{x-\sqrt{x^2+1}}$

Theorem (error estimate for bisection method)

Suppose f is continuous on $[a,b]$, and $f(a)\cdot f(b)<0$. Let $f(\alpha)=0$, $\alpha \in (a,b)$. Let $\{x_n\}$ be the sequence generated by the bisection method. Then

$$|\alpha - x_n| \le \frac{|b-a|}{2^{n+1}} \quad \text{for all } n \geq 0$$

Proof



$$|\alpha - x_n| \le \frac{|b_n - a_n|}{2} = \frac{1}{2^2}|b_{n-1} - a_{n-1}| = \frac{1}{2^3}|b_{n-2} - a_{n-2}| = \dots$$

$$\dots = \frac{1}{2^{n+1}}|b_0 - a_0| = \frac{|b-a|}{2^{n+1}}$$

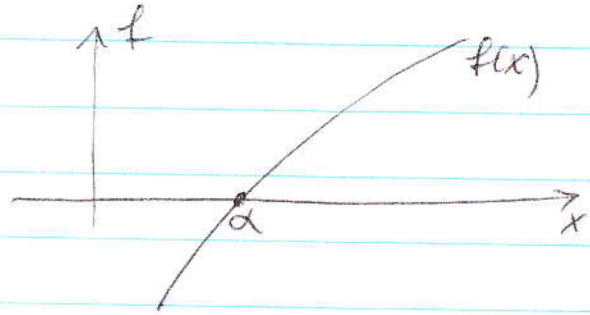
$\boxtimes$

Fixed-point iteration

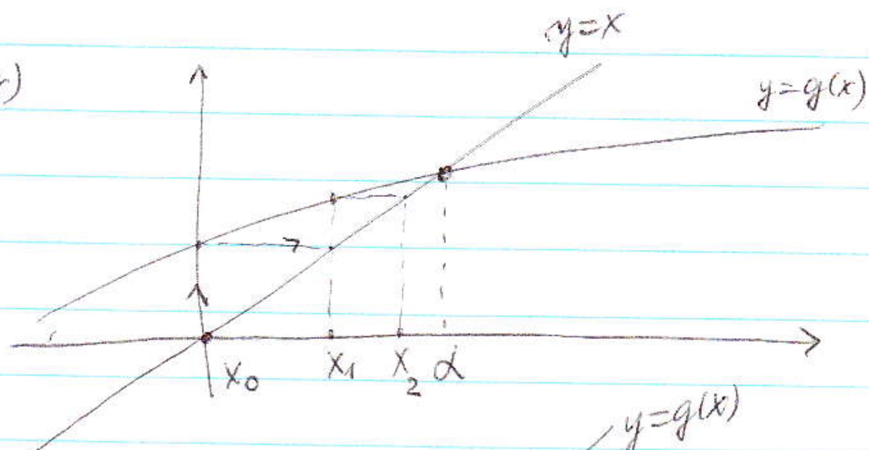
Suppose that $f(x)=0$ is equivalent to $x=g(x)$
We say that α is a fixed point of g $\Leftrightarrow$
α is a root of $f(x)$.

We define an iterative scheme
$$x_{n+1} = g(x_n), \text{ given } x_0.$$

$f(x)=0$

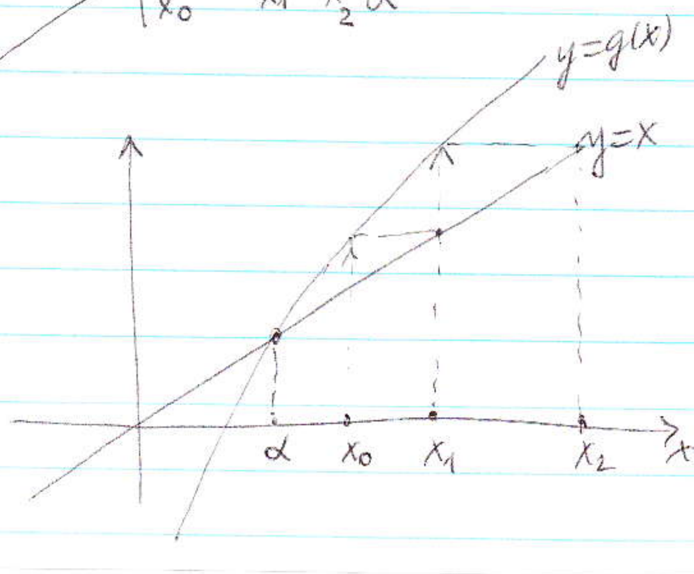


$x=g(x)$



converges

$x=g(x)$



diverges

Ex $f(x) = x^2 - 3$ $\qquad \alpha = \sqrt{3} = 1.73205$

$$g_1(x) = x - \left(\frac{x^2-3}{2}\right)$$

$f(x) = 0$

$C f(x) = 0 \quad | + x$

$C f(x) + x = x$

or

$x = \underbrace{C f(x) + x}_{g_1(x)}$

$C = -\frac{1}{2}$

n	x_n	
0	1.5	
1	1.875	
2	1.617	
3	1.810	
4	1.672	
5	1.774	converges

$$g_2(x) = \frac{3}{x}$$

$x^2 - 3 = 0$

$x^2 = 3 \quad | \frac{1}{x}$

$x = \underbrace{\frac{3}{x}}_{g_2(x)}$

n	x_n	
0	$1.5 = \frac{3}{2}$	
1	2	
2	1.5	
3	2	
4	1.5	diverges

Question

What condition on $g(x)$ guarantees that $x_n \to \alpha$?

Note

$x^2 - 3 = 0$

$x^2 = 3 \quad | + x^2$

$2x^2 = 3 + x^2 \quad | \frac{1}{2x}$

$x = \frac{1}{2x}(3 + x^2) = \overbrace{\frac{1}{2}\left(\frac{3}{x} + x\right)}^{= g_3(x)}$

Theorem (Existence of a unique fixed point)

A1 $g(x)$ maps $[a,b]$ into $[a,b]$, i.e. if $x \in [a,b]$ then $g(x) \in [a,b]$.

A2 $g(x)$ is continuous on $[a,b]$

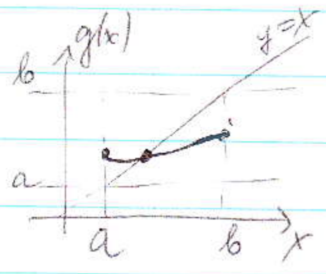
A3 $|g'(x)| \le k < 1$ for all $x \in [a,b]$

Note $A3 \Rightarrow A2$

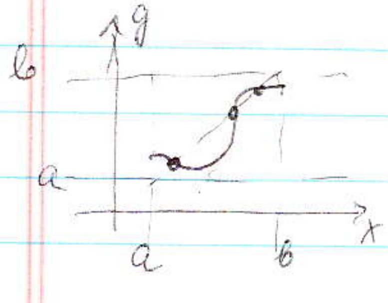
1) A1 and A2 are satisfied $\Rightarrow$ $g(x)$ has a fixed point $\alpha \in [a,b]$

2) A1 and A3 are satisfied $\Rightarrow$ fixed point α is unique.

Note



A1, A3 (and A2) are satisfied $\Rightarrow$ $g(x)$ has a unique fixed point



A3 fails $\Rightarrow$ 3 fixed points
A1 & A2 hold

HW #1 discussion

#4

$$Df = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

$$f' - Df = O(h^2), \text{ i.e. } \underbrace{f' - Df}_{\text{error}} = C \cdot h^2$$

1. $$\frac{f' - Df}{h^2} \sim C$$

slope = 2 = order of scheme

2. $$\underbrace{\ln \text{error}}_{y} = 2 \underbrace{\ln h}_{x} + \ln C$$

3. $$h \to \frac{h}{2} \Rightarrow \text{error} \to \frac{\text{error}}{4}$$