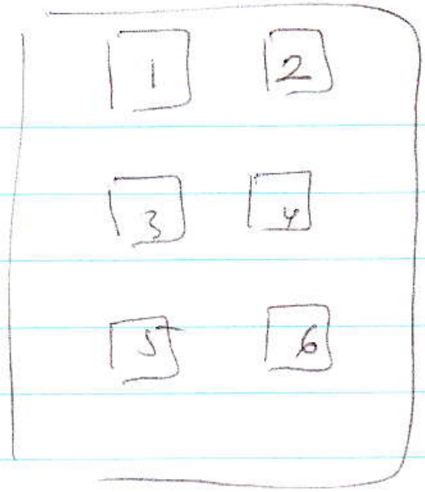
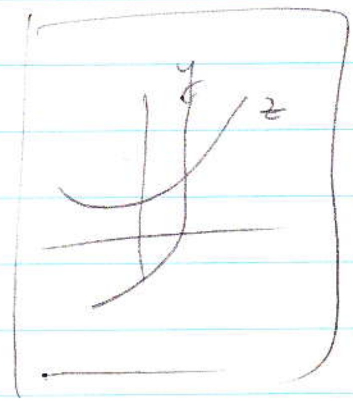
1/27/2010

subplot(3,2,1), plot(x,y)

subplot (3, 2, 2)



plot(x,y) and plot (x,z)

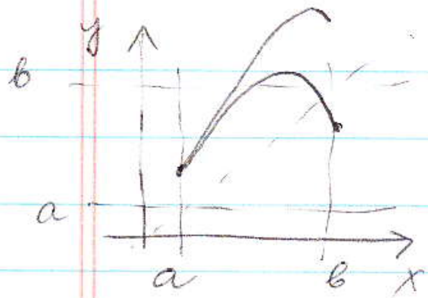


$\Rightarrow$ plot(x, y, x, z) or plot (x,y)
hold on
plot(x, z)

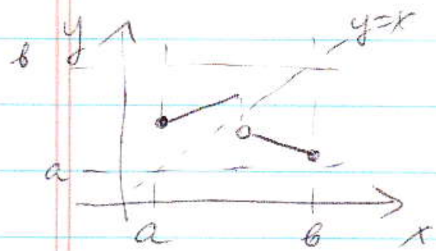
#4

$0.0000123 = 0.123 \times 10^{-4}$

0.001



A1 fails, there may or may not be a fixed point



A2 fails, A3 fails as well
$\Rightarrow$ there is no fixed point
A1 holds

Proof of (i)

$a \le x \le b$ If $g(a)=a$ or $g(b)=b$, then we are done.

Otherwise we suppose that $a < g(x) < b$ for $\forall x \in [a,b]$

$a \le g(x) \le b$ $a < g(a)$ and $g(b) < b$

Introduce $h(x) = x - g(x)$ is continuous since $g(x)$ is continuous.

$$h(a) = a - g(a) < 0, \qquad h(b) = b - g(b) > 0$$

By intermediate value theorem, there is $\alpha \in (a,b)$ such that $h(\alpha)=0 \Rightarrow h(\alpha) = \alpha - g(\alpha) = 0$

$$\Rightarrow \alpha = g(\alpha)$$

$\therefore$ α is a fixed point of g in (a,b).

OK

Proof of (2)

Suppose that α_1 and α_2 are two fixed points of $g(x)$ in $[a,b]$.

Then

$$|\alpha_1-\alpha_2| = |g(\alpha_1)-g(\alpha_2)| \overset{\text{Mean Value Thm}}{=} |g'(\xi)\cdot(\alpha_1-\alpha_2)|$$

$$= |g'(\xi)|\cdot|\alpha_1-\alpha_2| \le \underbrace{K}_{<1}\cdot|\alpha_1-\alpha_2|$$

$0<K<1$

$$\Rightarrow \overbrace{\underbrace{(1-K)}_{\neq 0}}^{>0}|\alpha_1-\alpha_2| \le 0 \Rightarrow |\alpha_1-\alpha_2|=0 \Rightarrow \alpha_1=\alpha_2 \ \lightning$$

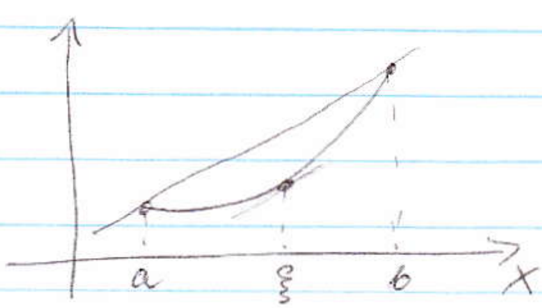
$\therefore$ fixed point is unique. ∎

Aside

Mean Value Th

Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a value $\xi \in (a,b)$ such that

$$f'(\xi) = \frac{f(b)-f(a)}{b-a} \Rightarrow f(b)-f(a) = f'(\xi)(b-a)$$



tangent line of $f(x)$ at ξ has the same slope as secant line connecting $(a, f(a))$ and $(b, f(b))$

Thm (Convergence of fixed-point iterations)

A_1 and A_3 hold $\Rightarrow$ the sequence defined by $x_{n+1}=g(x_n)$ converges for any $x_0 \in [a,b]$

Proof

$$|\alpha - x_{n+1}| = |g(\alpha) - g(x_n)| \overset{MVT}{=} |g'(\xi)| \cdot |\alpha - x_n| \le K \cdot |\alpha - x_n|$$

$$\therefore \; |\alpha - x_{n+1}| \le K \cdot |\alpha - x_n| \le K^2 |\alpha - x_{n-1}| \le K^3 |\alpha - x_{n-2}| \le \ldots$$

$$\ldots \le K^{n+1} |\alpha - x_0|$$

As $n \to \infty$, $K^{n+1} \to 0$ since $K<1 \Rightarrow$

$$|\alpha - x_{n+1}| \le K^{n+1} \underbrace{|\alpha - x_0|}_{\text{fixed}} \to 0 \text{ as } n \to \infty$$

$\Rightarrow$ $x_{n+1} \to \alpha$ as $n \to \infty$

Def of the order of convergence of a sequence

A sequence $\{x_n\}$ is said to converge to α with order r if there exists a constant C such that

$$|\alpha - x_{n+1}| \le C |\alpha - x_n|^r \quad \text{or}$$

$$|\alpha - x_n| \le C |\alpha - x_{n-1}|^r$$

Equivalent to

$$\frac{|\alpha - x_n|}{|\alpha - x_{n-1}|^r} \le C \quad \text{or} \quad \lim_{n \to \infty} \frac{|\alpha - x_n|}{|\alpha - x_{n-1}|^r} = \text{const}$$

r: order of convergence

C: asymptotic constant

Note 1. $|\alpha - x_n| \le K|\alpha - x_{n-1}|$: linear convergence

2. $K \sim |g'(\alpha)|$: we want to choose the iteration function $g(x)$ in such a way that $|g'(\alpha)|$ is as small as possible

Recall

$f(x) = x^2 - 3, \quad \alpha = \sqrt{3}$

$g_1(x) = x - \left(\frac{x^2-3}{2}\right) \qquad g_1'(x) = 1 - \frac{2x}{2} = 1 - x$

$|g_1'(\sqrt{3})| = 1 - \sqrt{3} = 0.73 < 1$

$g_2(x) = \frac{3}{x} \qquad g_2'(x) = -\frac{3}{x^2}$

$|g_2'(\sqrt{3})| = \frac{3}{(\sqrt{3})^2} = 1$