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How to determine the order of convergence numerically?

Def Sequence $\{x_n\}$ converges to α with order r if

$$\underbrace{|\alpha - x_n|}_{\text{error at iteration } n} \le C \cdot \underbrace{|\alpha - x_{n-1}|^r}_{\text{error}^r \text{ at iteration } n-1}$$

Denote $E_n = |\alpha - x_n|$

$$E_n \le C \cdot E_{n-1}^r \quad \Big/ \ln$$

$$\ln E_n \approx \ln C + r \ln E_{n-1} \quad \Big/ \frac{1}{\ln E_{n-1}}$$

$$\frac{\ln E_n}{\ln E_{n-1}} \approx r + \frac{\ln C}{\ln E_{n-1}}$$

Assume that C is close to 1 $\Rightarrow \ln C \approx 0$

$$\Rightarrow \boxed{\frac{\ln E_n}{\ln E_{n-1}} \approx r}$$

In practice, exact value α may not be known. Can we use this result to find order r?

Yes, if we note that

$$\alpha - x_n \approx x_{n+1} - x_n \Rightarrow \boxed{\frac{\ln |x_{n+1} - x_n|}{\ln |x_n - x_{n-1}|} \approx r}$$

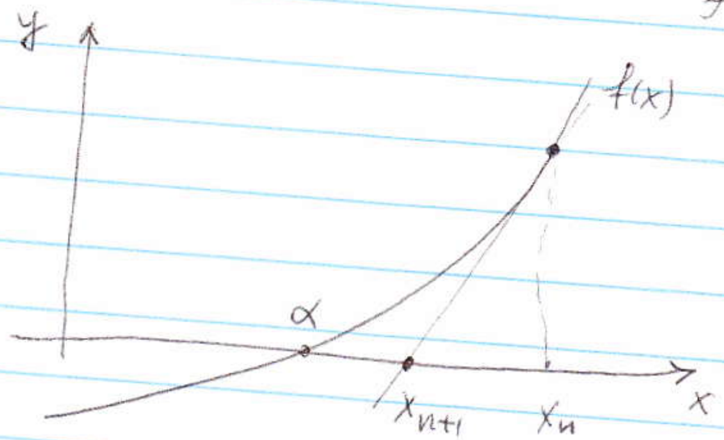
$f(x)=0$ Newton's method (local linear approximation)

Idea: expand $f(x_{n+1})$ around the point $x=x_n$

$$\overset{0}{f(x_{n+1})} \overset{\text{Taylor}}{=} f(x_n) + f'(x_n)(x_{n+1}-x_n) + \dots$$

$$f(x_{n+1}) = f(x_n + (x_{n+1}-x_n))$$

Then solve for $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, given x_0



$$\text{Slope} = f'(x_n) = \frac{f(x_n)-0}{x_n - x_{n+1}} \Rightarrow \text{as above}$$

Ex $f(x) = x^2 - 3 \Rightarrow$

$$x_{n+1} = \underbrace{x_n - \frac{f(x_n)}{f'(x_n)}}_{g(x_n)} = x_n - \frac{x_n^2-3}{2x_n}$$

$f'(x) = 2x$

n	x_n	$f(x_n)$	$\|\alpha - x_n\|$
0	1.5	-0.75	0.232051
1	1.75	0.0625	0.017949
2	1.7321429	0.000319	0.000092
3	1.7320509	0.0000001	0.0000001

Note

1. $x_{n+1} = g(x_n)$ where $g(x) = x - \frac{f(x)}{f'(x)}$

If $f(\alpha)=0$, $f'(\alpha)\neq 0$ (α is a simple root of f)

$$\Rightarrow \boxed{g'(\alpha)=0, \quad g''(\alpha)\neq 0}$$

$$g(x) = x - \frac{f(x)}{f'(x)} \Rightarrow g'(x) = 1 - \frac{1}{(f'(x))^2}\left[f'(x)\cdot f'(x) - f''(x)\cdot f(x)\right]$$

$$g'(x) = \frac{\cancel{(f'(x))^2} - \cancel{(f'(x))^2} + f(x)\cdot f''(x)}{[f'(x)]^2} = \frac{f(x)\,f''(x)}{(f'(x))^2}$$

$$g'(\alpha) = \frac{\overbrace{f(\alpha)}^{0}\,f''(\alpha)}{\underbrace{(f'(\alpha))^2}_{\neq 0}} = 0$$

It turnes out that $g''(\alpha) = \frac{f''(\alpha)}{f'(\alpha)} \neq 0$

2. It can be shown that if α is a simple root of $f(x)$, then

$$|\alpha - x_n| \leq C \cdot |\alpha - x_{n-1}|^2 : \text{ 2nd order convergent}$$

If α is a multiple root with multiplicity $m \geq 2$, then

$$|\alpha - x_n| \leq C |\alpha - x_{n-1}| : \text{ 1st order convergent}$$

3. Newton's method is more expensive than bisection, fixed-point (in general) since we have two function evaluation ($f(x_n)$, $f'(x_n)$) per iteration.

<u>thm</u> <u>Convergence of Newton's method</u>

Suppose function $f \in C^2[a,b]$ (i.e. f has continuous second derivatives) and assume that f has a simple root $\alpha \in (a,b)$, i.e. $f(\alpha)=0$, $f'(\alpha) \neq 0$.

Then Newton's method converges to α if x_0 is chosen sufficiently close to α.

<u>Proof</u>

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = g(x_n)$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

We showed that

$$g'(\alpha) = \left.\frac{f(x)\, f''(x)}{\left(f'(x)\right)^2}\right|_{x=\alpha} = 0$$

$g'(x)$ is continuous since $f(x)$, $f'(x)$ and $f''(x)$ are continuous. Therefore, there exists a small interval containing α on which $|g'(x)| \le k < 1$, i.e. $g'(x)$ is close to 0, since $g'(x)$ is continuous and $g'(\alpha) = 0$. By fixed point convergence theorem, sequence $\{x_n\}$ obtained using Newton's method will converge to α that is chosen sufficiently close to α, i.e. x_0 has to be inside the above interval

<u>Thm</u> <u>(Rate of convergence of Newton's method)</u>

Suppose that $f \in C^2[a,b]$, $f(\alpha) = 0$, $f'(\alpha) \neq 0$, x_0 is sufficiently close to α. Then Newton's method converges <u>quadratically</u>, i.e.

$$|\alpha - x_{n+1}| \le C\, |\alpha - x_n|^2$$

<u>Proof</u> Expand $f(\alpha)$ in the neighborhood of $x = x_n$

$$\frac{1}{f'(x_n)} \,\Big|\; 0 = f(\alpha) = f(x_n) + f'(x_n)(\alpha - x_n) + \frac{f''(\xi)}{2!}(\alpha - x_n)^2$$

$f(\alpha)$ $\parallel$ $x_n + (\alpha - x_n)$

where ξ is between α and x_n

$$0 = \frac{f(x_n)}{f'(x_n)} + (\alpha - x_n) + \frac{f''(\xi)}{2f'(x_n)}(\alpha - x_n)^2$$

Newton's method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$\Rightarrow \frac{f(x_n)}{f'(x_n)} = x_n - x_{n+1}$$

$$0 = (\cancel{x_n} - x_{n+1}) + (\alpha - \cancel{x_n}) + \frac{f''(\xi)}{2f'(x_n)}(\alpha - x_n)^2$$

$$0 = (\alpha - x_{n+1}) + \frac{f''(\xi)}{2f'(x_n)}(\alpha - x_n)^2$$

$$\Rightarrow \alpha - x_{n+1} = -\frac{f''(\xi)}{2f'(x_n)}(\alpha - x_n)^2$$

$|\alpha - x_{n+1}| \leq C \cdot |\alpha - x_n|^2$ where $C = \max\limits_{x \in [a,b]} \dfrac{|f''(\xi)|}{2|f'(x_n)|}$

or $C = \dfrac{\max |f''(\xi)|}{2 \min |f'(x_n)|}$