2/1/2010

#4

$x = 0.0001852 \qquad y = 0.0003899$

Wrong way: $\quad x = 0.0002 \qquad y = 0.0004$

$\Rightarrow x + y = 0.000 = 0.6 \times 10^{-3} = z_1$

Correct way: use floating point representation

$x = 0.1852 \times 10^{-3} \qquad y = 0.3899 \times 10^{-3}$

$x + y = (0.1852 + 0.3899) \times 10^{-3} =$

$= \underbrace{0.5651 \times 10^{-3}} = z_2$

$z_1 \neq z_2$!

Newton's method (Cont'd)

Last time we showed that

provided $f'(\alpha)\neq 0$

α: exact

$$\alpha - x_{n+1} = -\frac{1}{2}\frac{f''(\xi_n)}{f'(x_n)}(\alpha - x_n)^2 \approx$$

$$\approx -\frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}(\alpha - x_n)^2 = M(\alpha - x_n)^2$$

where $\boxed{M = -\frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}}$,

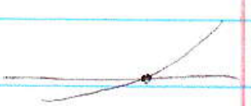
$$\Rightarrow \alpha - x_{n+1} = M\cdot(\alpha - x_n)^2 \quad \Big|\, M$$

$$M(\alpha - x_{n+1}) = \left(M(\alpha - x_n)\right)^2 = \left[M(\alpha - x_{n-1})\right]^{2^2} = \dots$$

$$\dots = \left[M(\alpha - x_0)\right]^{2^{n+1}}$$

α: root, x_0: initial guess, M: const

For convergence we need $|M(\alpha - x_0)| < 1$

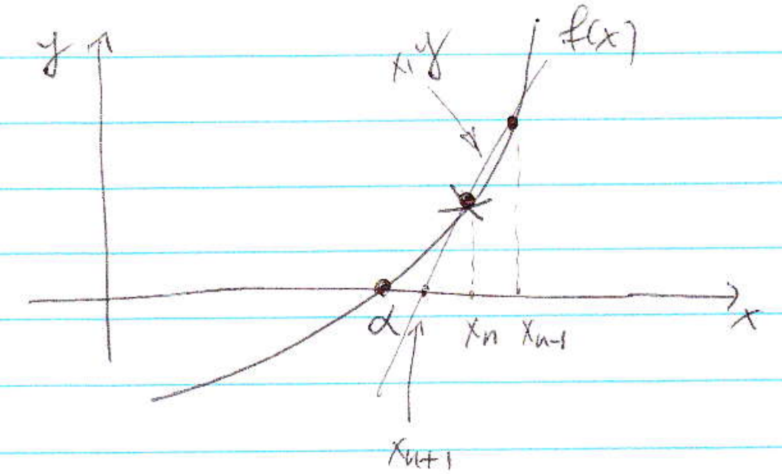


If $|M|$ is large, then x_0 may need to be chosen very closely to root α for convergence.

Secant method

$$x_{n+1} = x_n - \frac{f(x_n)}{\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}} \qquad \text{given } x_0, x_1$$

Geometrically



Slope of secant line: $\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$

Equation of secant line:

$y = ax + b$

$$\frac{y - f(x_n)}{x - x_n} = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

$y = kx + b$

$y - y_0 = k(x - x_0)$

$\frac{y - y_0}{x - x_0} = k$

Set $y = 0$ and $x = x_{n+1}$ $\Rightarrow$ solve for x_{n+1}: as above

Note

1. It can be shown that $|\alpha - x_n| \le C \cdot |\alpha - x_{n+1}|^p$

where $p = \frac{1+\sqrt{5}}{2} \sim 1.6$ golden ratio

superlinear convergence

2. Secant method converges more slowly than Newton's method but faster than fixed-point iterations (in general).

Summary

1. Bisection
   - linear convergence
   - guaranteed to converge
   - do not need $f'(x)$

2. Secant
   - superlinear convergence
   - convergence depends on initial guesses x_0 and x_1
   - do not need $f'(x)$

3. Newton's method
   - converges quadratically (if root α is simple)
   - convergence depends on x_0
   - need to know $f'(x)$

Linear systems of equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\dots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

n equations, n unknowns $x_1, x_2, \dots, x_n$

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad i = 1, \dots, n$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{b}$$

$$\Rightarrow Ax = b, \qquad A = (a_{ij})$$

Thm

Let A be an $n \times n$ matrix. The following conditions on A are equivalent:

1. The equation $Ax = b$ has a unique solution for any b.

2. $\det A \neq 0$

3. Matrix A is invertible, i.e. there is matrix that we denote A^{-1} such that $AA^{-1} = A^{-1}A = I$
   I: identity matrix
   A^{-1}: inverse matrix

4. Columns of A are linearly independent

5. Rows of A are linearly independent

6. Eigenvalues of A are all nonzero.

7. The equation $Ax = 0$ has a unique solution $x = 0$

Direct methods

$Ux = b$ where U is upper triangular matrix

$$u_{11}x_1 + u_{12}x_2 + \dots + u_{1n}x_n = b_1$$
$$u_{22}x_2 + \dots + u_{2n}x_n = b_2$$
$$- - - -$$
$$u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n = b_{n-1}$$
$$u_{nn}x_n = b_n$$

Then

$$x_n = b_n / u_{nn}$$

$$x_{n-1} = \left(b_{n-1} - u_{n-1,n}x_n\right) / u_{n-1,n-1}$$

$$- - - -$$

$$x_1 = \left(b_1 - \left(u_{12}x_2 + \dots + u_{1n}x_n\right)\right) / u_{11}$$

This is called <u>back substitution</u>.

Operation count

\# divisions $:= n$

\# multiplications $= n(n-1)/2$

Pf

$$\# \text{mult} = 1+2+\dots+n-1 = \sum_{k=1}^{n-1} k = S$$

$$2S = \sum_{k=1}^{n-1} k + \sum_{k=1}^{n-1} (n-k) = \sum_{k=1}^{n-1} \left(k+(n-k)\right) = \sum_{k=1}^{n-1} n$$

$$= n(n-1)$$

OK

The leading order term in operation count is $\frac{n^2}{2}$