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#1

$$\underbrace{\ldots E_2^l E_1^l}_{A} \underbrace{B}_{B} \underbrace{E_1^r \ldots}_{C}$$

Pivoting (Cont'd)

General case

Suppose A is invertible, but some pivot $a_{kk}^{(k)} = 0$

$$A^{(k)} = E_{k-1} \ldots E_2 E_1 A = \begin{pmatrix} a_{11}^{(1)} & \ldots & a_{1k}^{(1)} & \ldots & a_{1n}^{(1)} \\ & \ddots & & & \\ & & a_{kk}^{(k)} & \ldots & a_{kn}^{(k)} \\ & 0 & \vdots & & \\ & & a_{nk}^{(k)} & \ldots & a_{nn}^{(k)} \end{pmatrix}$$

$$\det A^{(k)} = \det E_{k-1} \cdot \det E_{k-2} \ldots \det E_2 \cdot \det E_1 \cdot \det A$$

$$= 1 \cdot 1 \ldots 1 \cdot 1 \cdot \underbrace{\det A}_{\neq 0} \neq 0$$

$E = \begin{pmatrix} 1 & 0 \\ \| & 1 \end{pmatrix}$

$\det E = 1$

$\Rightarrow \det A^{(k)} \neq 0 \Rightarrow A^{(k)}$ is also invertible

If $a_{ik}^{(k)} = 0$, $i = k+1, \ldots, n$, then one can show that the first k columns of $A^{(k)}$ are linearly dependent which contradicts the fact that $A^{(k)}$ is nonsingular. Let i be the first row in which $a_{ik}^{(k)} \neq 0$. Then switch rows k and i and proceed.

$i = k+1, \ldots, n$

$$A^{(k)} \longrightarrow P_k A^{(k)}, \quad P_k: \text{permutation matrix}$$

$$
P_k = \begin{array}{cl}
\begin{array}{ccccccccccc} & & & \text{col. } k & & & & \text{column } i & & & \end{array} & \\
\left(\begin{array}{ccccccccccc}
1 & & & & & & & & & & \\
& \ddots & & & & & & & & & \\
& & 1 & & & & & & & & \\
& & & 0 & \cdots & \cdots & \cdots & 1 & & & \\
& & & \vdots & 1 & & & \vdots & & & \\
& & & \vdots & & \ddots & & \vdots & & & \\
& & & \vdots & & & 1 & \vdots & & & \\
& & & 1 & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots \\
& & & & & & & & 1 & & \\
& & & & & & & & & \ddots & \\
& & & & & & & & & & 1
\end{array}\right) &
\begin{array}{l} \\ \\ \\ \text{row } k \\ \\ \\ \\ \text{row } i \\ \\ \\ \\ \end{array}
\end{array}
$$

<u>Note</u> $\det P_k = -1$

<u>Thm</u> Let A be nonsingular. Then there exists a permutation matrix P such that $PA = LU$ where L is unit lower triangular matrix, U is upper triangular matrix.

<u>Pf</u> We can pivot to produce

$$E_{n-1} P_{n-1} \cdots E_2 P_2 E_1 P_1 A = U = \begin{pmatrix} a_{11}^{(1)} & \cdots & a_{1n}^{(1)} \\ & \ddots & \vdots \\ 0 & & a_{nn}^{(n)} \end{pmatrix}$$

some P_i may be identity matrices.

In general, $P_2 E_1 \neq E_1 P_2$. But there exists another transformation matrix $\tilde{E}_1$ such that

$$P_2 E_1 = \tilde{E}_1 P_2$$

$PA = LU$

Then

$$\underbrace{\tilde{E}_{n-1} \dots \tilde{E}_2 \tilde{E}_1}_{} \underbrace{P_{n-1} \dots P_2 P_1}_{P} A = U$$

$$P_{n-1} \dots P_2 P_1 A = \left(\tilde{E}_{n-1} \dots \tilde{E}_2 \tilde{E}_1\right)^{-1} U$$

$$\underbrace{P_{n-1} \dots P_2 P_1}_{P} A = \underbrace{\tilde{E}_1^{-1} \tilde{E}_2^{-1} \dots \tilde{E}_{n-1}^{-1}}_{L} \cdot U$$

$$L = \begin{pmatrix} 1 & & & \\ * & 1 & & 0 \\ * & * & \ddots & \\ & & & 1 \end{pmatrix}$$

$\Rightarrow PA = LU$ ∎

Note

1. P contains pivoting information: in practice, it is not necessary to store matrix P, the information can be stored in the integer pivotal vector / array:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{matrix} p(1) = 3 \\ p(2) = 1 \\ p(3) = 2 \end{matrix}$$

2. Since $\det P_k = \pm 1 \Rightarrow \det A = \pm a_{11}^{(1)} a_{22}^{(2)} \dots a_{nn}^{(n)}$

Def Matrix A is called strictly diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \qquad i=1,\dots,n$$

Def Matrix A is called positive definite if
$x^T A x > 0$ for all $x \neq 0$

3. $A_k = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix}$: $k \times k$ leading submatrix of A

Thm
If matrix A is strictly diagonally dominant or A is positive definite or determinants of all leading submatrices are ($\det A_k \neq 0$) all nonzero, then pivots arising in Gaussian elimination are nonzero and pivoting is not needed, i.e. $A = LU$

Note In practice, pivoting is recommended when pivot is small.