2/17/2010

$A_h u = f$: linear system of equations

A_h : tridiagonal, symmetric

Questions

1. How to solve $A_h u = f$ efficiently?

2. Is matrix A_h invertible for all h, $c(x)$?

3. Does the method converge?, i.e.

$$\lim_{h \to 0} \max_i |\phi_i - u_i| = 0 \, ?$$

if so, what is the order of accuracy?

$$\max_i |\phi_i - u_i| = O(h^p)$$

LU factorization of a tridiagonal matrix

$$\begin{pmatrix} b_1 & c_1 & & & & 0 \\ a_2 & b_2 & c_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & a_n & b_n \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} 1 & & & & 0 \\ l_2 & 1 & & & \\ & l_3 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & l_n & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} v_1 & c_1 & & & 0 \\ & v_2 & c_2 & & \\ & & \ddots & \ddots & \\ 0 & & & v_{n-1} & c_{n-1} \\ & & & & v_n \end{pmatrix}}_{U}$$

To determine L, U:

$b_1 = v_1 \quad \Rightarrow \quad v_1 = b_1$

$$\left.\begin{array}{lll} a_k = l_k \cdot v_{k-1} & \Rightarrow & l_k = a_k / v_{k-1} \\ b_k = l_k \cdot c_{k-1} + v_k & \Rightarrow & v_k = b_k - l_k \cdot c_{k-1} \end{array}\right\} k = 2, .., n$$

$Ax = y$

$\underbrace{LUx}= f$

To solve $Ly = f$

$$\begin{pmatrix} 1 & & & & 0 \\ l_2 & 1 & & & \\ & l_3 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & l_n & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{pmatrix}$$

$y_1 = f_1$ $\qquad\qquad$ $y_1 = f_1$

$l_k y_{k-1} + y_k = f_k \Rightarrow y_k = f_k - l_k y_{k-1}, \quad k = 2, \dots, n$

To solve $Uu = y$

$$\begin{pmatrix} v_1 & c_1 & & & \\ & v_2 & c_2 & & \\ & & \ddots & \ddots & \\ & & & v_{n-1} & c_{n-1} \\ & & & & v_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}$$

$v_n u_n = y_n \Rightarrow u_n = y_n / v_n$

$v_k u_k + c_k \cdot u_{k+1} = y_k \Rightarrow u_k = (y_k - c_k u_{k+1}) / v_k$

$k = n-1, \dots, 1$

Operation count: $\#\text{mult} \sim 3n \ll \frac{n^3}{3}$ (HW)

Recall

A is positive definite if $x^T A x > 0$ for $x \neq 0$.

Ex $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ is positive definite

Proof

$$(x_1 \ \ x_2) \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 & -x_1 + 2x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (2x_1 - x_2)x_1 + (-x_1 + 2x_2)x_2 = 2x_1^2 - x_1x_2 - x_1x_2 + 2x_2^2$$

$$= 2(x_1^2 + x_2^2) - 2x_1x_2 = (x_1^2 + x_2^2 - 2x_1x_2) + x_1^2 + x_2^2$$

$$= (x_1 - x_2)^2 + x_1^2 + x_2^2 \geq 0$$

If $x^T \neq (0, 0)$, then $x^T A x > 0$

Ex $A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$ is NOT positive definite

Proof

$$x^T A x = x_1^2 + x_2^2 - 4x_1x_2$$

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow x^T A x = 1^2 + 0^2 - 0 = 1 > 0$$

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow x^T A x = 1^2 + 1^2 - 4 = -2 < 0$$

OK

Claim:

If matrix A is positive definite, then A is invertible.

Pf $Ax=0 \Rightarrow x^TAx=0$, A is positive definite

$\Rightarrow x=0 \Rightarrow A$ is invertible

(Note A is invertible if $Ax=0$ has the only solution $x=0$)

Claim

If $c(x)\geq 0$, then matrix A_h arising in the finite difference approximation scheme is positive definite (and invertible) for all $h>0$.

Pf
$$x^TA_hx = \sum_{i,j=1}^{n} a_{ij}x_ix_j = \frac{1}{h^2}\Big\{\sum_{i=1}^{n}(2+\underbrace{c_ih^2}_{\geq 0})x_i^2 -$$

$$-\sum_{i=1}^{n-1}x_ix_{i+1} - \sum_{i=2}^{n}x_ix_{i-1}\Big\} \geq$$

$$\geq \frac{1}{h^2}\Big\{\sum_{i=1}^{n}2x_i^2 - 2\sum_{i=1}^{n-1}x_ix_{i+1}\Big\} =$$

$$= \frac{1}{h^2}\Big\{x_1^2 + x_n^2 + \sum_{i=1}^{n-1}x_i^2 + \sum_{i=1}^{n-1}x_{i+1}^2 - 2\sum_{i=1}^{n-1}x_ix_{i+1}\Big\} =$$

$$= \frac{1}{h^2} \left\{ x_1^2 + x_n^2 + \sum_{i=1}^{n-1} (x_i - x_{i+1})^2 \right\} \geq 0$$

If $x^T A x = 0 \Rightarrow$ $x_1^2 = 0$

$x_n^2 = 0$

$(x_i - x_{i+1})^2 = 0$, $i = 1, .., n-1$

$\Rightarrow x_1 = x_2 = \dots = x_n = 0$

$\Rightarrow x^T A x = 0$ iff $x = 0$ $\Rightarrow$ A is positive definite.

OK