Convergence

If $c(x) \geq 0$, then $\max_i |u_i - \phi_i| \leq \frac{1}{96} |\max \phi^{(4)}(x)| \cdot h^2$

Error analysis

$AX = b$, A: invertible

x: exact solution, $x = A^{-1}b$
$\tilde{x}$: approximate solution
$e = x - \tilde{x}$: error
$r = b - A\tilde{x}$: residual

Claim: $Ae = r$

Pf

$Ae = A(x - \tilde{x}) = Ax - A\tilde{x} = b - A\tilde{x} = r$

Note

$e = 0 \Leftrightarrow r = 0$

However, small residual r doesn't always imply that error e is also small

Def a vector norm $\|x\|$ has the following properties

1. $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$
2. $\|\alpha x\| = |\alpha| \cdot \|x\|$ where α is a scalar
3. $\|x + y\| \leq \|x\| + \|y\|$: triangle inequality

Ex Let $x = (x_1, \dots, x_n)^T$.

$\|x\|_\infty = \max\{|x_i|, \ i=1,\dots,n\}$ $\quad l_\infty$ norm

$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$ $\quad l_2$ norm

$\|x\|_1 = \sum_{i=1}^{n} |x_i|$ $\quad l_1$ norm

Ex $x = (1,1,1)$ $\quad \|x\|_\infty = 1, \quad \|x\|_2 = \sqrt{3}, \quad \|x\|_1 = 3$

Check that $\|x\|_\infty$ is indeed a norm

Pf

1. Clearly $\|x\|_\infty \geq 0$

$\|x\|_\infty = 0 \Leftrightarrow \max_i |x_i| = 0 \Leftrightarrow |x_i| = 0 \Leftrightarrow x_i = 0$

$\Leftrightarrow x = 0$

2. $\|\alpha x\|_\infty = \max_i \{|\alpha x_i|, \ i=1,\dots,n\} =$

$= |\alpha| \cdot \max\{|x_i|, \ i=1,\dots,n\} = |\alpha| \cdot \|x\|_\infty$

3. $\|x+y\|_\infty = \max\{|x_i + y_i|, \ i=1,\dots,n\} \leq$

$\leq \max\{|x_i| + |y_i|, \ i=1,\dots,n\} \leq$

$\leq \max\{|x_i|, i=1,\dots,n\} + \max\{|y_i|, \ i=1,\dots,n\} =$

$= \|x\|_\infty + \|y\|_\infty$

Ex
$$\left(\begin{array}{cc|c} \multicolumn{2}{c}{\overbrace{\hphantom{1.01 \quad 0.99}}^{A}} & \\ 1.01 & 0.99 & 2 \\ 0.99 & 1.01 & 2 \end{array}\right) \qquad x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : \text{exact solution}$$

$$b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\tilde{x}_1 = \begin{pmatrix} 1.01 \\ 1.01 \end{pmatrix} \qquad e_1 = x - \tilde{x}_1 = \begin{pmatrix} -0.01 \\ -0.01 \end{pmatrix} \qquad r_1 = b - A\tilde{x}_1 = \begin{pmatrix} -0.02 \\ -0.02 \end{pmatrix}$$

$$\tilde{x}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad e_2 = x - \tilde{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad r_2 = b - A\tilde{x}_2 = \begin{pmatrix} -0.02 \\ 0.02 \end{pmatrix}$$

Thus $\|e_1\|_\infty = 0.01, \quad \|e_2\|_\infty = 1$

$$\|r_1\|_\infty = \|r_2\|_\infty = 0.02$$

$\|x\|_\infty = 1$

$\|b\|_\infty = 2$

$$\frac{\|e_1\|_\infty}{\|x\|_\infty} = \frac{\|r_1\|_\infty}{\|b\|_\infty} \quad \text{but} \quad \frac{\|e_2\|_\infty}{\|x\|_\infty} = 100 \cdot \frac{\|r_2\|_\infty}{\|b\|_\infty}$$

$$\frac{0.01}{1} \qquad \underset{0.01}{\underset{\parallel}{\frac{0.02}{2}}} \qquad \frac{1}{1} \qquad \overset{\shortparallel}{\frac{0.02}{2}} = 0.01$$

Q What is the relation between relative error $\frac{\|e\|}{\|x\|}$ and relative residual $\frac{\|r\|}{\|b\|}$?

Def A matrix norm $\|A\|$ has the following properties:

1. $\|A\| \geq 0$ and $\|A\| = 0 \Leftrightarrow A = 0$
2. $\|\alpha A\| = |\alpha| \cdot \|A\|$, where α is a scalar
3. $\|A+B\| \leq \|A\| + \|B\|$
4. $\|A \cdot B\| \leq \|A\| \cdot \|B\|$

Given a vector norm $\|x\|$, the subordinate (or induced) matrix norm is defined by

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} \qquad \left(\|A\| = \max_{\|x\|=1} \|Ax\| \right)$$

The subordinate matrix norm has an additional property:

5. $\|Ax\| \leq \|A\| \cdot \|x\|$ for all vectors x.

Ex $A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}$ $\quad \|A\|_\infty = \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = 6$

(proof later)

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad Ax_1 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \frac{\|Ax_1\|_\infty}{\|x_1\|_\infty} = \frac{4}{1} = 4$$

$$x_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad Ax_2 = \begin{pmatrix} 5 \\ 5 \\ 1 \end{pmatrix} \Rightarrow \qquad \frac{\|Ax_2\|_\infty}{\|x_2\|_\infty} = \frac{5}{1} = 5$$

$$x_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad Ax_3 = \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix} \Rightarrow \qquad \frac{\|Ax_3\|_\infty}{\|x_3\|_\infty} = \frac{6}{1} = 6$$

<u>Claim</u> $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$ ("max row sum")