Claim $\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$ "max row sum"

Pf

By definition, $\|A\|_\infty = \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}$

Step 1

$$|(Ax)_i| = \left|\sum_{j=1}^n a_{ij}\cdot x_j\right| \le \sum_{j=1}^n |a_{ij}|\cdot|x_j| \le \|x\|_\infty \sum_{j=1}^n |a_{ij}|$$

i^{th} component

$$\Rightarrow |(Ax)_i| \le \|x\|_\infty \max_i \sum_{j=1}^n |a_{ij}|, \quad \forall i$$

$$\|Ax\|_\infty \le \|x\|_\infty \cdot \max_i \sum_{j=1}^n |a_{ij}|$$

$$\frac{\|Ax\|_\infty}{\|x\|_\infty} \le \max_i \sum_{j=1}^n |a_{ij}| \quad \text{for } \forall x \neq 0$$

$$\|A\|_\infty \le \max_i \sum_{j=1}^n |a_{ij}|$$

Step 2

Suppose $\max_i \sum_{j=1}^n |a_{ij}| = \sum_{j=1}^n |a_{pj}|$

Define $y = (y_1, \ldots, y_n)^T$ with $y_j = \begin{cases} 1 & \text{if } a_{pj} \ge 0 \\ -1 & \text{if } a_{pj} < 0 \end{cases}$

$$\|A\|_\infty = \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \geq \frac{\|Ay\|_\infty}{\|y\|_\infty} = \|Ay\|_\infty \geq$$

$$\geq |(Ay)_p| = \left|\sum_{j=1}^{n} a_{pj} y_j\right| = \sum_{j=1}^{n} |a_{pj}| = \max_i \sum_{j=1}^{n} |a_{ij}|$$

$$\Rightarrow \|A\|_\infty \geq \max_i \sum_{j=1}^{n} |a_{ij}|$$

Combining two steps we get $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$

<u>Note</u>

1. $\|x\|_2 = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2} = \left(x^T \cdot x\right)^{1/2}$

$$\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_i \{ \sqrt{\lambda_i} \text{ where}$$

λ_i are eigenvalues of $A^TA \}$

<u>Def</u> λ is an eigenvalue of matrix A with associated eigenvector $x \neq 0$ if $Ax = \lambda x$

e'values can be computed by solving characteristic eq^n

$$\det(A - \lambda I) = 0$$

2. $\|x\|_1 = \sum_{j=1}^{n} |x_j|$

$$\|A\|_1 = \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_j \sum_{i=1}^{n} |a_{ij}|$$

"max column sum"

<u>Thm</u>

Let A be invertible, $Ax = b$

x: exact solution, $\tilde{x}$: approximation

$e = x - \tilde{x}$, $r = b - A\tilde{x}$: residual
error

Then
$$\frac{\|e\|}{\|x\|} \leq K(A) \cdot \frac{\|r\|}{\|b\|}$$

where $K(A) = \|A\| \cdot \|A^{-1}\|$: condition number of matrix A relative to norm $\|.\|$

<u>Proof</u>

$$\|b\| = \|Ax\| \leq \|A\| \cdot \|x\| \Rightarrow \frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|}$$

$$Ae = r \Rightarrow e = A^{-1} r$$

$$\|e\| = \|A^{-1} r\| \leq \|A^{-1}\| \cdot \|r\|$$

$$\Rightarrow \frac{\|e\|}{\|x\|} \leq \overbrace{\frac{\|A\|}{\|b\|} \cdot \|A^{-1}\|}^{= K(A)} \|r\| \Rightarrow \frac{\|e\|}{\|x\|} \leq K(A) \cdot \frac{\|r\|}{\|b\|}$$

$\blacksquare$

Recall

$$A = \begin{pmatrix} 1.01 & 0.99 \\ 0.99 & 1.01 \end{pmatrix} \qquad \|A\|_\infty = 2$$

$$A^{-1} = \begin{pmatrix} 25.25 & -24.75 \\ -24.75 & 25.25 \end{pmatrix} \qquad \|A^{-1}\|_\infty = 50$$

$$\Rightarrow \underset{\infty}{K}(A) = \|A\|_\infty \cdot \|A^{-1}\| = 100$$

<u>Note</u>

1. $\begin{matrix} Ax = b \\ A\tilde{x} = \tilde{b} \end{matrix} \Rightarrow \dfrac{\|x - \tilde{x}\|}{\|x\|} \le K(A) \dfrac{\|b - \tilde{b}\|}{\|b\|}$

2. $\begin{matrix} Ax = b \\ \tilde{A}\tilde{x} = b \end{matrix} \Rightarrow \dfrac{\|x - \tilde{x}\|}{\|\tilde{x}\|} \le K(A) \dfrac{\|A - \tilde{A}\|}{\|A\|}$

Proof of 1 follows from Thm, proof of 2 — HW

<u>Recall</u> $\qquad \varepsilon = 10^{-2} \Rightarrow K(A) = 4.0404$

$$A = \begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix} \qquad \underset{\infty}{K}(A) \sim 4$$

$$A^{(1)} = E_1 A = \begin{pmatrix} \varepsilon & 1 \\ 0 & 1 - \frac{1}{\varepsilon} \end{pmatrix} \qquad K_\infty(E_1 A) \sim \frac{1}{\varepsilon^2} \qquad \tilde{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$E_1 P_1 A = \begin{pmatrix} 1 & 1 \\ 0 & 1-\varepsilon \end{pmatrix} \qquad K_\infty(E_1 P_1 A) \sim 4 \qquad \tilde{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Gaussian elimination is unstable because $\kappa(A^{(k)}) >> \kappa(A)$. Perturbations of $A^{(k)}$ due to roundoff errors are amplified by $\kappa(A^{(k)})$ instead of $\kappa(A)$.

Gaussian elimination with partial pivoting is stable because $\kappa(A^{(k)}) \sim \kappa(A)$.

Iterative methods

$$Ax = b \iff x = Bx + C$$

$$x_{k+1} = Bx_k + C$$

B: iteration matrix
C: constant vector