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Error analysis

Def scheme has order of convergence r if

$$\lim_{n\to\infty} \frac{|\alpha - x_{n+1}|}{|\alpha - x_n|^r} = C \qquad \text{or} \qquad \frac{E_{n+1}}{E_n^{\,r}} \sim C$$

$$E_{n+1} = C E_n^r$$

$$\boxed{\ln E_{n+1} = \ln C + r \ln E_n}$$

$$\boxed{\text{if } C \approx 1} \Rightarrow \quad r \approx \frac{\ln E_{n+1}}{\ln E_n}$$

Iterative Methods

$$Ax = b \iff x = Bx + c$$

$$x_{k+1} = Bx_k + c$$

B: iteration matrix

Jacobi's method (simultaneous displacements)

$A = L + D + U$: matrix splitting

$D = \text{diag}(a_{11}, a_{22}, .., a_{nn})$, assume $a_{ii} \neq 0$ $\quad i = 1, .., n$

$$L = \begin{pmatrix} 0 & & & 0 \\ a_{21} & 0 & & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & a_{n,n-1} & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & 0 & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ 0 & & & 0 \end{pmatrix}$$

$$Ax = b \iff (L+D+U)x = b$$

$$\iff Dx = -(L+U)x + b \quad | \; D^{-1}$$

$$\iff x = -D^{-1}(L+U)x + D^{-1}b$$

$B_J = -D^{-1}(L+U)$: iteration matrix for Jacobi's method

$$D^{-1} = \mathrm{diag}\left(\frac{1}{a_{11}}, \frac{1}{a_{22}}, \ldots, \frac{1}{a_{nn}}\right)$$

$Dx_{k+1} = -(L+U)x_k + b$: easy to solve for x_{k+1}

<u>Components</u>

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$a_{11} x_1^{(k+1)} = -a_{12} x_2^{(k)} - a_{13} x_3^{(k)} + b_1$$

$$a_{22} x_2^{(k+1)} = -a_{21} x_1^{(k)} - a_{23} x_3^{(k)} + b_2$$

$$a_{33} x_3^{(k+1)} = -a_{31} x_1^{(k)} - a_{32} x_2^{(k)} + b_3$$

In general,

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right]$$

Ex

$2x_1 - x_2 = 1 \qquad x_1 = 1$

$-x_1 + 2x_2 = 1 \qquad x_2 = 1$

$$2 x_1^{(k+1)} = x_2^{(k)} + 1$$

$$2 x_2^{(k+1)} = x_1^{(k)} + 1$$

k	$x_1^{(k)}$	$x_2^{(k)}$	
0	0	0	arbitrary guess
1	$\frac{1}{2}$	$\frac{1}{2}$	
2	$\frac{3}{4}$	$\frac{3}{4}$	
3	$\frac{7}{8}$	$\frac{7}{8}$	converges to $\binom{1}{1}$

Define $e_k = x - x_k$: <u>error</u>

Then $\|e_0\|_\infty = 1$

$$\|e_1\|_\infty = \frac{1}{2}$$

$$\|e_2\|_\infty = \frac{1}{4}$$

$$\|e_3\|_\infty = \frac{1}{8}$$

$\Rightarrow \|e_{k+1}\|_\infty = \frac{1}{2}\|e_k\|_\infty$ for $k \geq 0$

$$Ax = b$$
$$x = Bx + c$$
$$x_{k+1} = Bx_k + c$$
$e_k = x - x_k$: error

<u>Thm</u> If $\|B\| < 1$ for some subordinate matrix norm then $x_k \to x$ for any initial guess x_0.

<u>Proof</u> $e_k = x - x_k = Bx + c - (Bx_{k-1} + c) = B(x - x_{k-1}) =$
$= Be_{k-1}$

$$e_k = Be_{k-1} = B^2 e_{k-2} = B^3 e_{k-3} = \ldots = B^k e_0$$

$$\|e_k\| = \|B^k e_0\| \leq \|B^k\| \cdot \|e_0\|$$

$\|B^2\| = \|B \cdot B\| \leq \|B\| \cdot \|B\| = \|B\|^2$

$\Rightarrow \|B^k\| \leq \|B\|^k$

$\Rightarrow \|e_k\| \leq \|B^k\| \cdot \|e_0\| \leq \|B\|^k \cdot \|e_0\| \to 0$ as $k \to \infty$

since $\|B\| < 1$

$\Rightarrow \|e_k\| \to 0$ as $k \to \infty$ $\Rightarrow \|x - x_k\| \to 0$

$\Rightarrow x_k \to x$ as $k \to \infty$

<u>Important</u>:

$e_{k+1} = B e_k \Rightarrow \|e_{k+1}\| \leq \|B\| \cdot \|e_k\|$

<u>Ex</u>

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad B_J = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

<u>Note</u> $\|B_J\|_\infty = \frac{1}{2} < 1$

$\Rightarrow$ we know that Jacobi's method converges.