Gauss-Seidel method (successive displacements)

$A = L + D + U$: as before

$Ax = b \iff (L + D + U)x = b$

$$(L+D)x = -Ux + b$$

$(L+D)x_{k+1} = -Ux_k + b$: easy to solve for x_{k+1}

$B_{GS} = -(L+D)^{-1}U$ iteration matrix for Gauss-Seidel method

Components

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$a_{11}x_1^{(k+1)} = -a_{12}x_2^{(k)} - a_{13}x_3^{(k)} + b_1$$

$$a_{22}x_2^{(k+1)} = -a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} + b_2$$

$$a_{33}x_3^{(k+1)} = -a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)} + b_3$$

To compute $x_i^{(k+1)}$ we use already updated components $x_1^{(k+1)}, \dots, x_{i-1}^{(k+1)}$

We use $x_i^{(k+1)}$ as soon as it becomes available

$$Ax=b \iff x = Bx + c$$

$$x_{k+1} = Bx_k + c, \quad \text{given } x_0$$

$x_1^{(k+1)}$

$$x = (x_1, \dots, x_n)$$

$$x_{k+1} = \left(x_1^{(k+1)}, \dots, x_n^{(k+1)}\right)$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right]$$

<u>Ex</u>

$$2x_1 - x_2 = 1$$

$$-x_1 + 2x_2 = 1$$

$$\begin{aligned} 2x_1^{(k+1)} &= 1 + x_2^{(k)} \\ 2x_2^{(k+1)} &= 1 + x_1^{(k+1)} \end{aligned} \implies \begin{aligned} x_1^{(k+1)} &= \frac{1}{2}\left(1 + x_2^{(k)}\right) \\ x_2^{(k+1)} &= \frac{1}{2}\left(1 + x_1^{(k+1)}\right) \end{aligned}$$

k	$x_1^{(k)}$	$x_2^{(k)}$
0	0	0
1	$\frac{1}{2}$	$\frac{3}{4}$
2	$\frac{7}{8}$	$\frac{15}{16}$
3	$31/32$	$63/64$ – converges faster than Jacobi

$\|e_0\|_\infty = 1$

$\|e_1\|_\infty = \frac{1}{2}$

$\|e_2\|_\infty = \frac{1}{8}$

$\|e_3\|_\infty = \frac{1}{32}$

In general, $\|e_{k+1}\|_\infty = \frac{1}{4}\|e_k\|_\infty, \quad k \geq 1$

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad B_{GS} = -(L+D)^{-1}U = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix}$$

$\|B_{GS}\|_\infty = \frac{1}{2} < 1 \Rightarrow$ G-S method converges

Summary

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$B_J = -D^{-1}(L+U) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \Rightarrow \|B_J\|_\infty = \frac{1}{2}$$

$$B_{GS} = -(L+D)^{-1}U = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix} \Rightarrow \|B_{GS}\|_\infty = \frac{1}{2}$$

In both cases, we proved that $\|e_{k+1}\|_\infty \leq \frac{1}{2}\|e_k\|_\infty$

In fact, we showed that

$$\|e_{k+1}\|_\infty = \frac{1}{2}\|e_k\|_\infty \quad \text{for Jacobi}$$

$$\|e_{k+1}\|_\infty = \frac{1}{4}\|e_k\|_\infty \quad \text{for Gauss-Seidel.}$$

Recall

Given A, $Ap = \lambda p$, $p \neq 0$.

λ is an eigenvalue with associated eigenvector p, $p \neq 0$, if $Ap = \lambda p$.

$$Ap = \lambda p \iff (A - \lambda I)p = 0_{p\neq 0} \iff \det(A - \lambda I) = 0$$

$f_A(\lambda) = \det(A - \lambda I)$: characteristic polynomial of A

Roots of $f_A(\lambda)$ are <u>eigenvalues</u> of A.

<u>Ex</u> If A is upper triangular matrix, then the eigenvalues are its diagonal elements.

<u>Pf</u>

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ & & \ddots & \vdots \\ & 0 & & a_{nn} \end{pmatrix}$$

$$f_A(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ & a_{22}-\lambda & \ddots & a_{n-1,n} \\ & 0 & \ddots & a_{nn}-\lambda \end{pmatrix} =$$

$$= (a_{11}-\lambda)(a_{22}-\lambda)\dots(a_{nn}-\lambda)$$

$$f_A(\lambda) = \det(A-\lambda I) = 0 \iff \lambda = a_{ii} \text{ for some } i = 1, \dots, n$$

$\blacksquare$

<u>Explanation</u> of $\|e_{k+1}\|_\infty = \frac{1}{4}\|e_k\|_\infty$ for G-S.

We showed

$$e_{k+1} = B e_k \qquad B = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix}$$

$\lambda_1 = 0$ is an e'value w/ associated e'vector $p_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = \frac{1}{4}$ is an e'value w/ e'vector $p_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

<u>Check</u> $\quad Bp = \lambda p$

$$Bp_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda_1 \cdot p_1$$

$$Bp_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Recall how to compute e'vectors.

$\lambda_2 = \frac{1}{4} \qquad Bp = \lambda p \iff (B - \lambda I)p = 0$

$$B - \lambda_2 \cdot I = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$$

Let $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$-\frac{1}{4} y_1 + \frac{1}{2} y_2 = 0$$

Let $y_2 = s$: parameter $\Rightarrow \frac{1}{4} y_1 = +\frac{1}{2} y_2 = +\frac{1}{2} s$

$y_1 = 2s$

$$\Rightarrow y = \begin{pmatrix} 2s \\ s \end{pmatrix} = s \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{p_2}$$